Entanglement Negativity in many-body physics

QMath13: Mathematical Results in Quantum Physics
(New mathematical topics arising in current theoretical physics)

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Motivation

- How to measure many-body entanglement?

$$\Psi(x_1, x_2, \cdots, x_N) = \sum_{\alpha} F_{\alpha}(x_1, x_2, \cdots, x_N)$$

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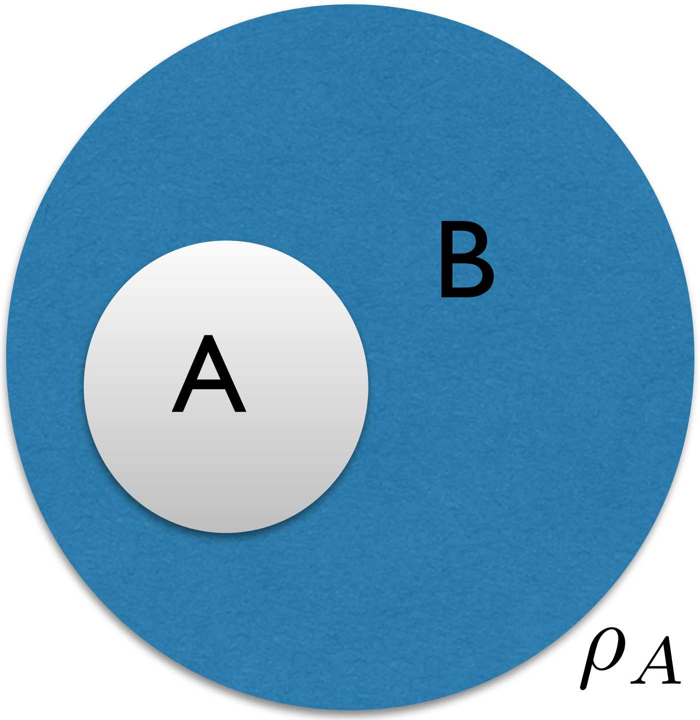
$$\Psi(x_1, x_2, \cdots, x_N) = \sum_\alpha F_\alpha(x_1, x_2, \cdots, x_N)$$

$\rho = |\Psi\rangle\langle\Psi|$



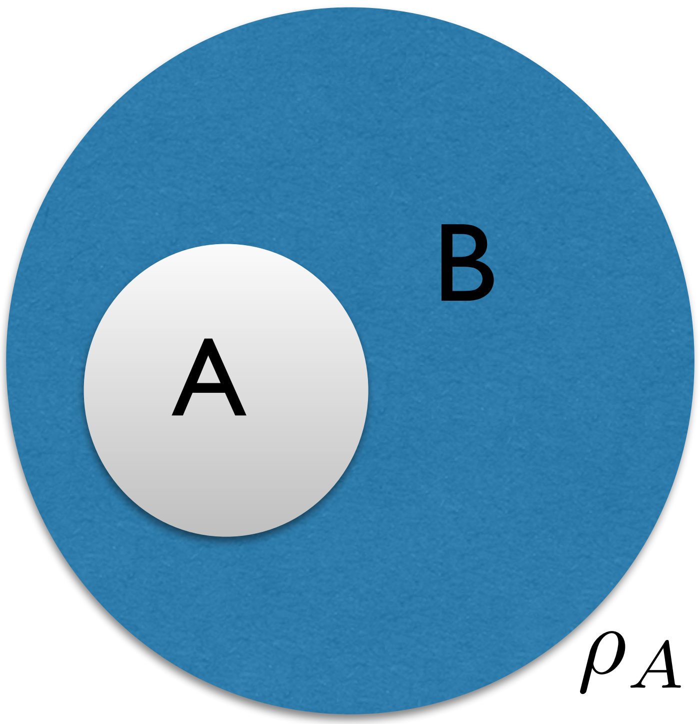
$\rho_A = \mathrm{Tr}_B \rho$

The bipartite entanglement measures the entanglement between A and B, which is the complementary part of A)

Entanglement entropy

$$S_A = -\mathrm{Tr}\rho_A \ln \rho_A = S_B$$

$$\rho = |\Psi\rangle\langle\Psi|$$



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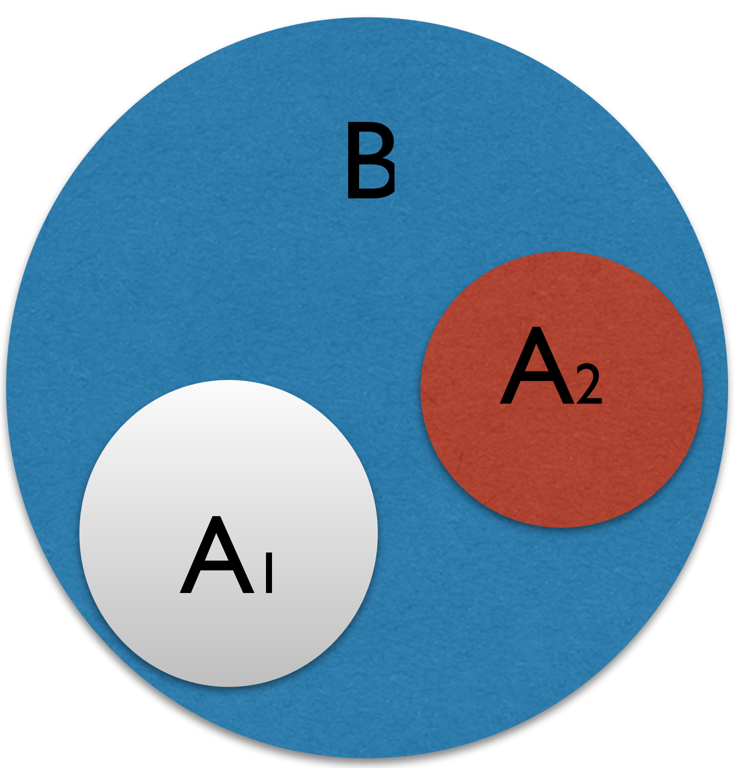


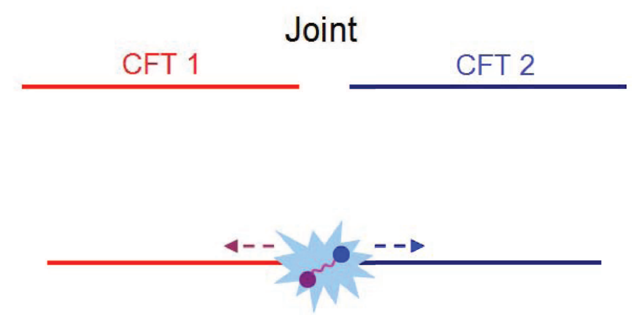
What is the entanglement between A1 and A2?

Entanglement negativity

[Peres, 1996,…]

My plan for today

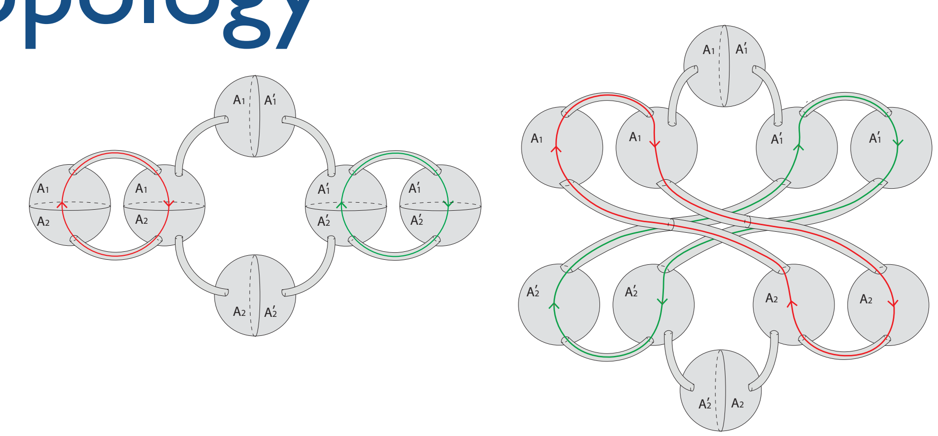
- Entanglement negativity for free fermion—hard to compute
  [**PYC**, X. Wen,16]
- Entanglement negativity for Conformal field theory—can measure the entanglement spread under quantum quenches



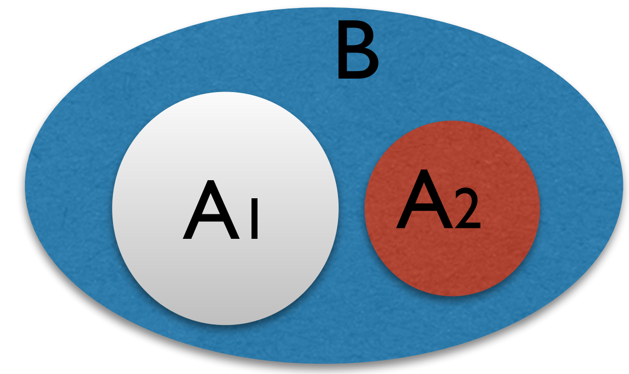
[X. Wen, **PYC**, S. Ryu,15]

- Entanglement negativity for Chern-Simon theories can relate to geometry and topology

[X. Wen, **PYC**, S. Ryu,16]



Entanglement negativity



$\rho_A = \rho_{A_1 \cup A_2}$ a mixed state
(after tracing out B)

Partial transpose:

$$\langle \phi_{A_1 i} \phi_{\boxed{A_2 j}} | \rho_A^{T_{A_2}} | \phi_{A_1 k} \phi_{\boxed{A_2 l}} \rangle = \langle \phi_{A_1 i} \phi_{\boxed{A_2 l}} | \rho_A | \phi_{A_1 k} \phi_{\boxed{A_2 j}} \rangle$$

$|\phi_{A_\alpha i}\rangle$ basis of $\mathcal{H}_{A_\alpha}$

Entanglement negativity:

$$\mathcal{E} := \ln \mathrm{Tr} |\rho_A^{T_{A_2}}| \qquad \mathrm{Tr}|\rho_A^{T_{A_2}}| = \sum_i |\lambda_i| = 1 - 2 \sum_{\lambda_i < 0} \lambda_i$$

measuring the negative eigenvalues of $\rho_A^{T_{A_2}}$

e.g., A entangled state $$|\Psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

$$\rho = |\Psi\rangle\langle\Psi| = \begin{array}{cccc} |1_A 0_B\rangle & |0_A 1_B\rangle & |0_A 0_B\rangle & |1_A 1_B\rangle \end{array} \atop \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$|10\rangle\langle 01| \rightarrow |11\rangle\langle 00|$

$$\rho^T = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\lambda_i = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}$$

$$\mathcal{E} = \ln \mathrm{Tr} |\rho^T| = \ln 2$$

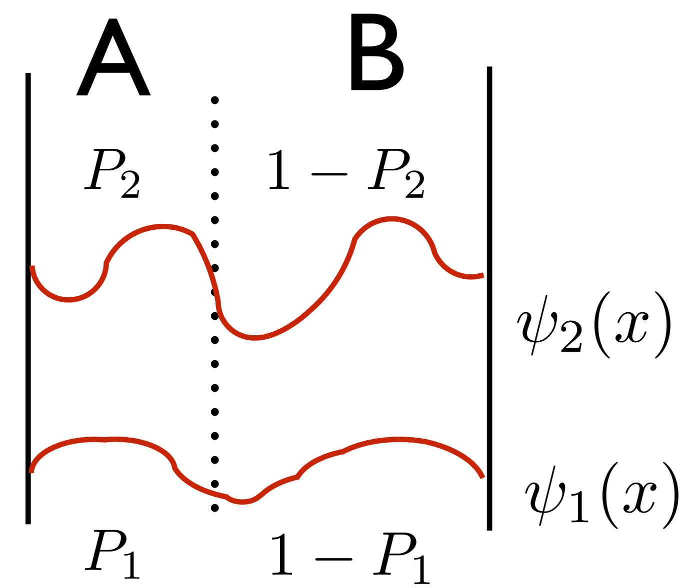
capture the entanglement!

Entanglement negativity for free fermions

[**PYC,** XW 2016]

A pure state (a bipartite system)

$$|\Psi\rangle = \prod_{i=1}^{N}(\sqrt{P_i}d^\dagger_{Ai} + \sqrt{1-P_i}d^\dagger_{Bi})|0\rangle$$



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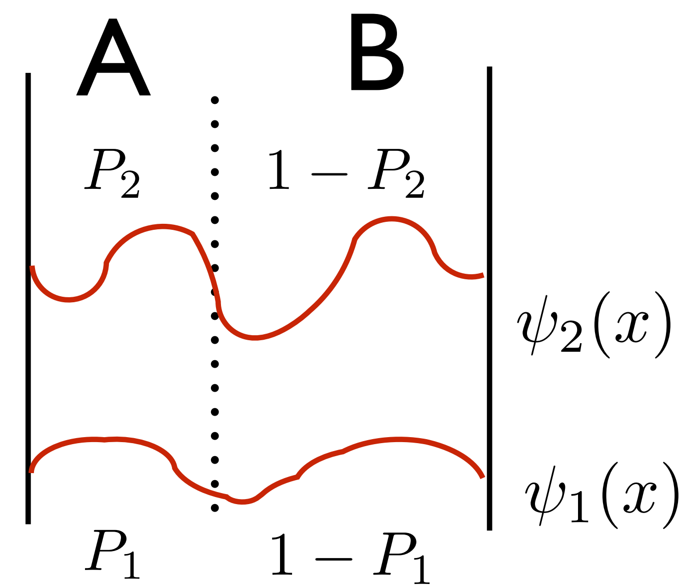

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$$\rho = \bigotimes_i \begin{array}{cc} |1_A 0_B\rangle & |0_A 1_B\rangle \\ \left(\begin{array}{c} P_i \\ \sqrt{P_i(1-P_i)} \end{array} \right. & \left. \begin{array}{c} \sqrt{P_i(1-P_i)} \\ 1-P_i \end{array} \right) \end{array}$$

$$|10\rangle\langle 01| \to |11\rangle\langle 00|$$

$$\rho^{T_B} = \bigotimes_i \rho_i^{T_B}$$

$$= \bigotimes_i \begin{array}{c} \begin{array}{cccc} |1_A 0_B\rangle & |0_A 1_B\rangle & |0_A 0_B\rangle & |1_A 1_B\rangle \end{array} \\ \begin{pmatrix} P_i & 0 & 0 & 0 \\ 0 & 1-P_i & 0 & 0 \\ 0 & 0 & 0 & \sqrt{P_i(1-P_i)} \\ 0 & 0 & \sqrt{P_i(1-P_i)} & 0 \end{pmatrix} \end{array}$$

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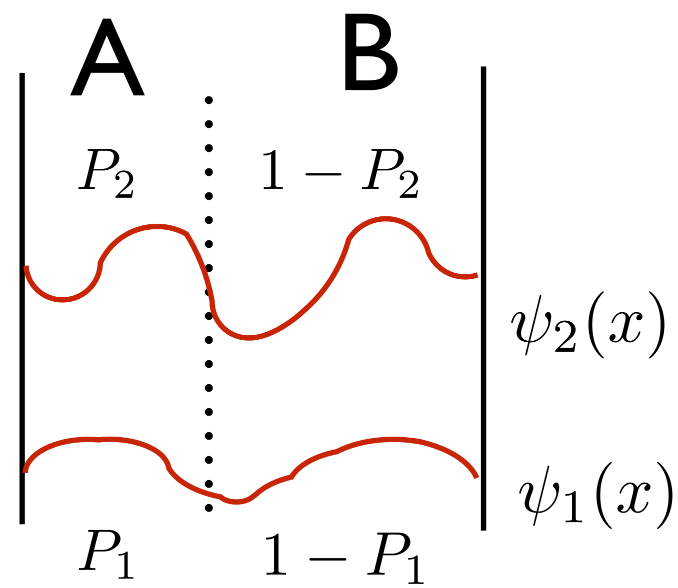

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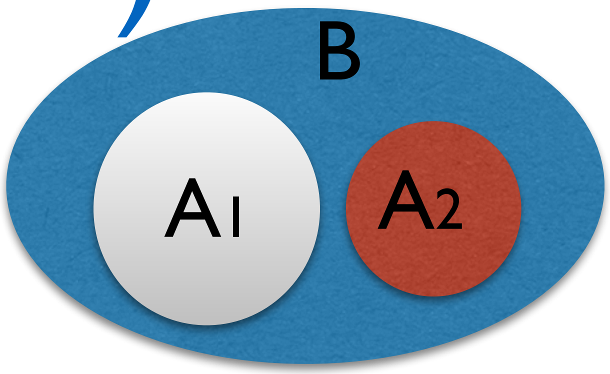
Entanglement negativity

$$\mathcal{E} = \ln\mathrm{Tr}|\rho^{T_B}| = \ln\prod_i\sum_\alpha(|\Xi_{i,\alpha}|)$$

$$= \sum_i \ln(1+2\sqrt{P_i(1-P_i)}).$$

Mixed states (a tripartite system)



The ground state *may not* be factored

$$|\Psi\rangle = \prod_{i=1}^{N}(\sqrt{P_i}\underbrace{\boxed{\sum_k V_{ik}(\sqrt{Q_k}d^{\dagger}_{A_1 k} + \sqrt{1-Q_k}d^{\dagger}_{A_2 k})}}_{d^{\dagger}_{Ai}} + \sqrt{1-P_i}d^{\dagger}_{Bi})|0\rangle,$$

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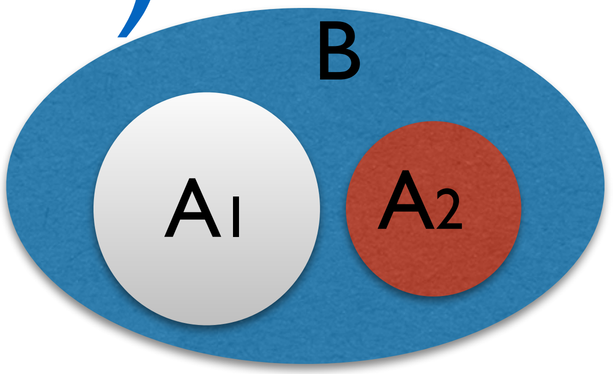

$$|\Psi\rangle = \prod_{i=1}^{N}\Big(\sqrt{P_i}\,\underbrace{\boxed{\sum_k V_{ik}\big(\sqrt{Q_k}\,d^\dagger_{A_1 k} + \sqrt{1-Q_k}\,d^\dagger_{A_2 k}\big)}}_{d^\dagger_{Ai}} + \sqrt{1-P_i}\,d^\dagger_{Bi}\Big)|0\rangle,$$

$$\rho_A = \sum_{n=0}^{N} D_n |n_A\rangle\langle n_A|$$

$V^{\mathrm{minor}}_{\{i\},\{k\}} :=$ picking $i_1, i_2, \cdots, i_n$-th rows and $k_1, k_2, \cdots, k_n$-th columns of V.

$$D_n|n_A\rangle\langle n_A| = \sum_{i_1>i_2>\cdots>i_n} P_{i_1}P_{i_2}\cdots P_{i_n} \prod_{\alpha\neq i_1,i_2,\cdots,i_n}(1-P_\alpha) \sum_{k_1>k_2>\cdots>k_n}\ \sum_{k'_1>k'_2>\cdots>k'_n} \mathrm{Det}[V^{\mathrm{minor}}_{\{i\},\{k\}}]\mathrm{Det}[V^{\mathrm{minor}}_{\{i\},\{k'\}}]$$
$$\times\big(\sqrt{Q_{k_1}}\,d^\dagger_{A_1k_1} + \sqrt{1-Q_{k_1}}\,d^\dagger_{A_2k_1}\big)\cdots\big(\sqrt{Q_{k_n}}\,d^\dagger_{A_1k_n} + \sqrt{1-Q_{k_n}}\,d^\dagger_{A_2k_n}\big)|0\rangle$$
$$\times\langle 0|\big(\sqrt{Q_{k'_1}}\,d_{A_1k'_1} + \sqrt{1-Q_{k'_1}}\,d_{A_2k'_1}\big)\cdots\big(\sqrt{Q_{k'_n}}\,d_{A_1k'_n} + \sqrt{1-Q_{k'_n}}\,d_{A_2k'_n}\big)$$

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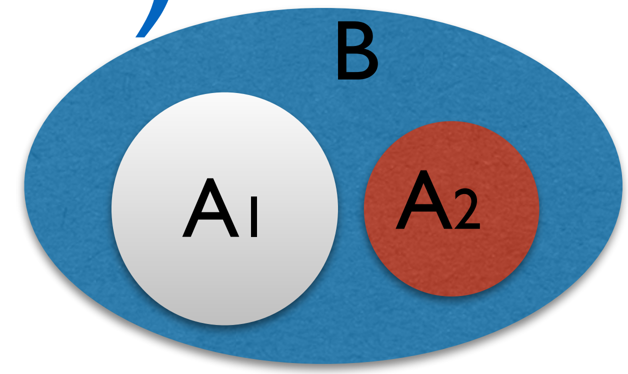

$|\Psi\rangle = \prod^{N}$ [redacted] $|0\rangle,$

Very hard to compute!!!

$\rho_A = \sum$ [redacted] -th rows

$D_n|n_A\rangle\langle n_A|$ [redacted] $V^{\text{minor}}_{\{i\},\{k'\}}]$

$i_1>i_2>\cdots>i_n \qquad \alpha\neq i_1,i_2,\cdots,i_n \qquad k_1>k_2>\cdots>k_n \; k'_1>k'_2>\cdots>k'_n$

$$\times(\sqrt{Q_{k_1}}d^\dagger_{A_1k_1}+\sqrt{1-Q_{k_1}}d^\dagger_{A_2k_1})\cdots(\sqrt{Q_{k_n}}d^\dagger_{A_1k_n}+\sqrt{1-Q_{k_n}}d^\dagger_{A_2k_n})|0\rangle$$

$$\times\langle 0|(\sqrt{Q_{k'_1}}d_{A_1k'_1}+\sqrt{1-Q_{k'_1}}d_{A_2k'_1})\cdots(\sqrt{Q_{k'_n}}d_{A_1k'_n}+\sqrt{1-Q_{k'_n}}d_{A_2k'_n})$$

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- Entanglement negativity for Chern-Simon theories can relate to geometry and topology

Recent development of computing entanglement negativity for a many body state!!!

- A replica trick + QFT (can be CFT or CS)

  [Calabrese, Cardy, Tonni, 12,13]
  [Wen, **P.-Y., Chang**, Ryu,15]

- Monte Carlo simulations

  [Chung, Alba, Bonnes, Chen, Lauchli,13]

- Tensor network (MPS)

  [Calabrese, Tagliacozzo, Tonni,13]

- An overlap matrix method (free fermions)

  [**P.-Y., Chang**, Wen,16]

- Representation theory (Valance bond solids)

  [Santos, Korepin,16]

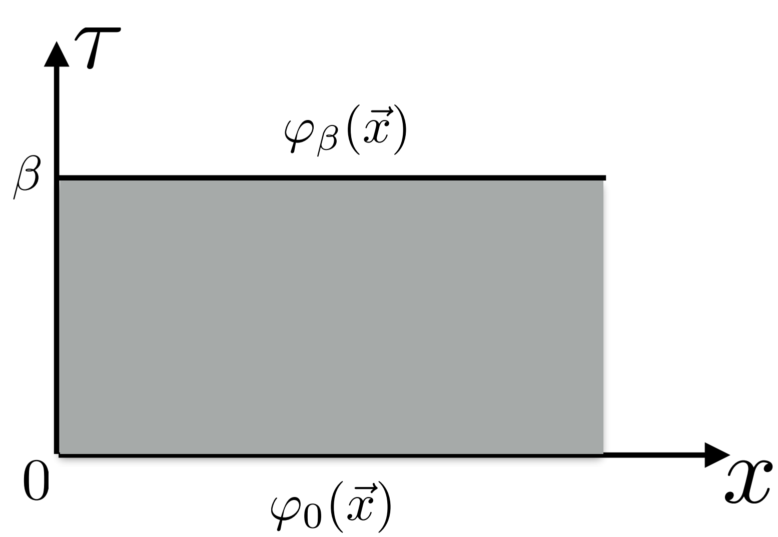
- A surgery method

  [Wen, **P.-Y., Chang**, Ryu,16]

A path integral representation and a replica trick

1. Density matrix

$$\rho\Big[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\Big] = \frac{1}{Z(\beta)}\,\big\langle\{\varphi_0(\vec{x})\}|e^{-\beta H}|\{\varphi_\beta(\vec{x})\}\big\rangle$$
$$= \int \prod [d\phi(\vec{x},\tau)] e^{-S_E} \prod_{\vec{x}} \delta[\phi(\vec{x},0)-\varphi_0(\vec{x})]\delta[\phi(\vec{x},\beta)-\varphi_\beta(\vec{x})]$$

$\rho = \dfrac{1}{Z}$ 

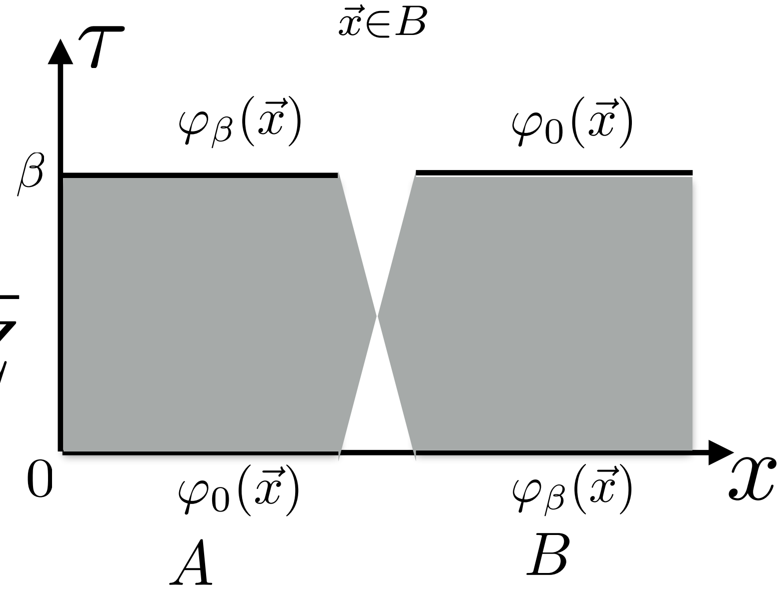
$Z =$ 

2. Partially transposed density matrix

$$\rho^{T_B}\Big[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\Big] = \int \prod_{\vec{x},\tau}[d\phi(\vec{x},\tau)]e^{-S_E}\prod_{\vec{x}\notin B}\delta[\phi(\vec{x},0)-\varphi_0(\vec{x})]\delta[\phi(\vec{x},\beta)-\varphi_\beta(\vec{x})]$$

$$\prod_{\vec{x}\in B}\delta[\phi(\vec{x},0)-\varphi_\beta(\vec{x})]\delta[\phi(\vec{x},\beta)-\varphi_0(\vec{x})].$$

$$\rho^{T_B} = \frac{1}{Z}$$



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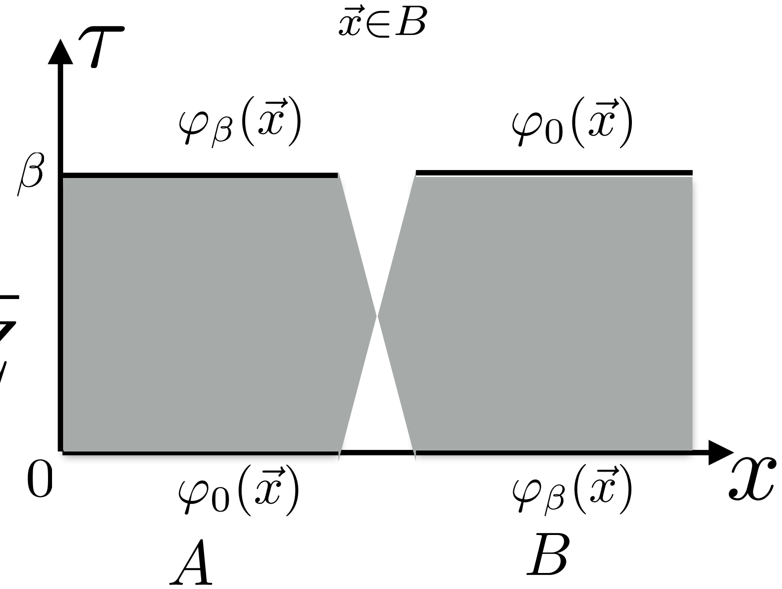
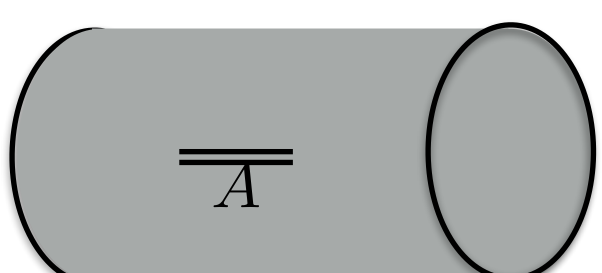


3. Reduced density matrix

$$\rho_{A_1\cup A_2}\Big[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\Big|\vec{x}\in A_1\cup A_2\Big]$$

$$= \int\left(\prod_{\vec{x}\in B}[d\varphi_0(\vec{x})d\varphi_\beta(\vec{x})]\delta[\varphi_0(\vec{x})-\varphi_\beta(\vec{x})]\right)\rho\Big[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\Big].$$

$$\rho_A = \frac{1}{Z}$$

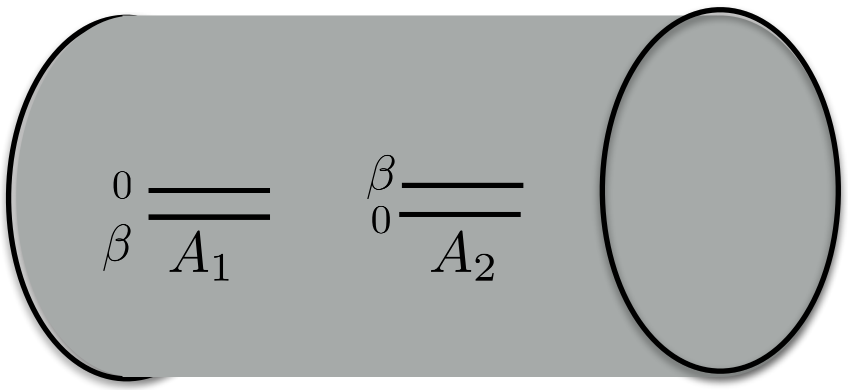


4. Partially transposed reduced density matrix

$$\rho_{A_1\cup A_2}^{T_{A_2}}\left[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\middle|\vec{x}\in A_1\cup A_2\right]$$
$$=\int\left(\prod_{\vec{x}\in B}[d\varphi_0(\vec{x})d\varphi_\beta(\vec{x})]\delta[\varphi_0(\vec{x})-\varphi_\beta(\vec{x})]\right)\rho^{T_{A_2}}\left[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\right].$$

Not easy to compute

$$\rho_A^{T_{A_2}}=\frac{1}{Z}$$

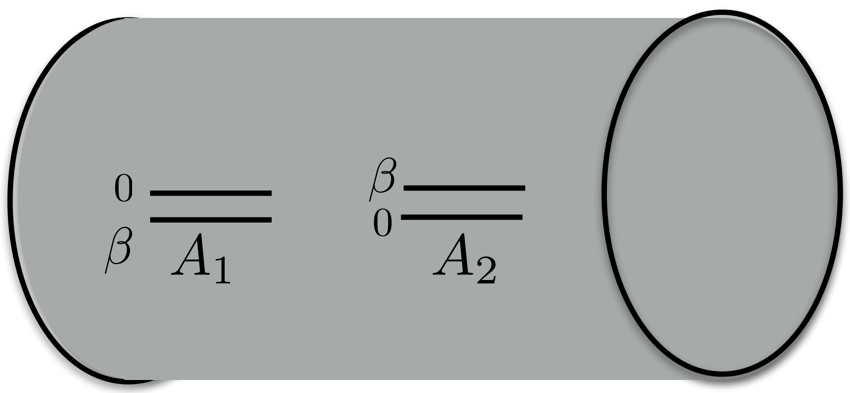


4. Partially transposed reduced density matrix

$$
\begin{aligned}
&\rho^{T_{A_2}}_{A_1\cup A_2}\left[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\middle|\vec{x}\in A_1\cup A_2\right] \\
&=\int\left(\prod_{\vec{x}\in B}[d\varphi_0(\vec{x})d\varphi_\beta(\vec{x})]\delta[\varphi_0(\vec{x})-\varphi_\beta(\vec{x})]\right)\rho^{T_{A_2}}\left[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\right].
\end{aligned}
$$

Not easy to compute

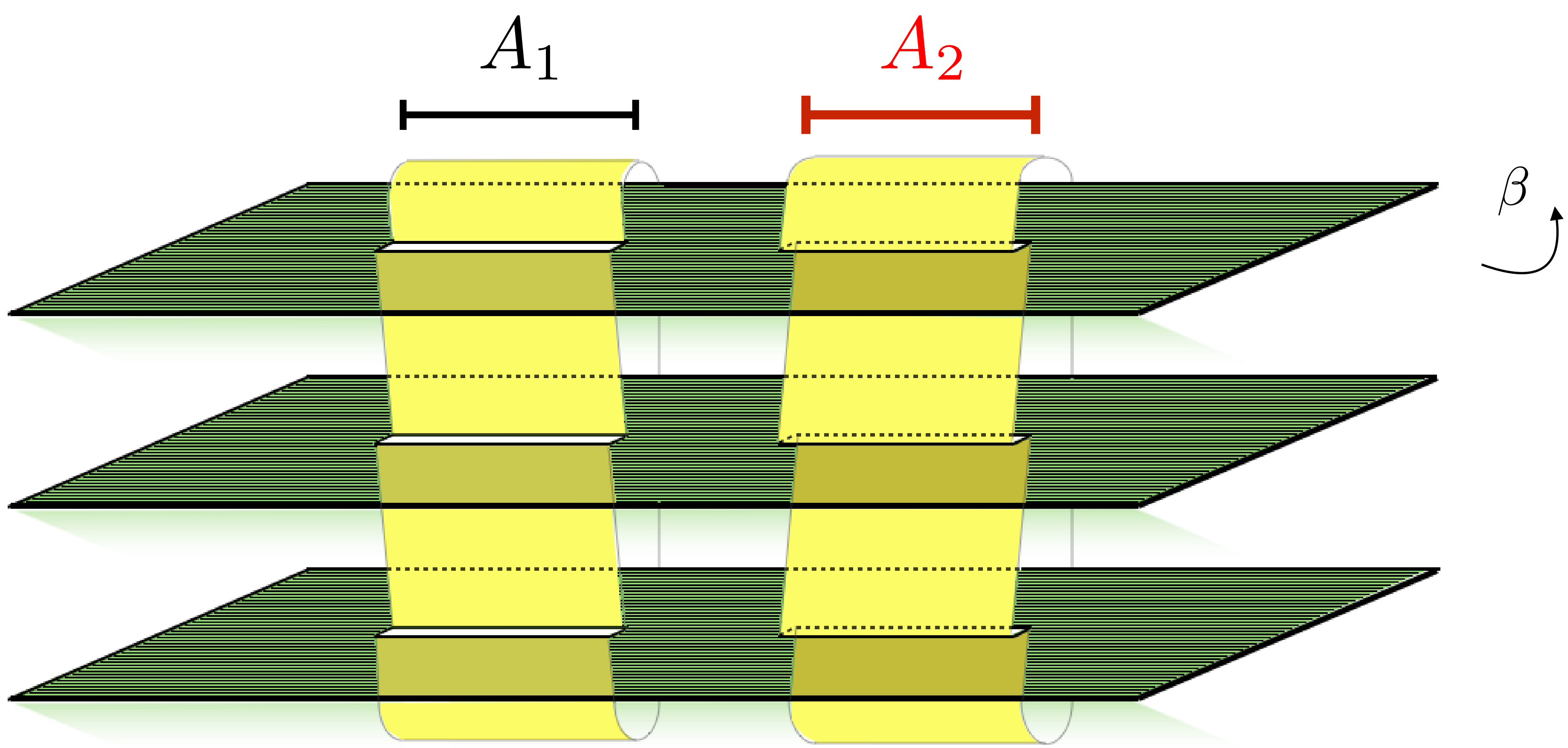
$$\rho_A^{T_{A_2}} = \frac{1}{Z}$$



5. Replica trick (n copies)

$$
\begin{aligned}
\mathrm{tr}\left(\rho^{T_{A_2}}_{A_1\cup A_2}\right)^n = \int\prod_{k=1}^{n}\Bigg\{&\prod_{\vec{x}}\left[d\varphi_0^{(k)}(\vec{x})d\varphi_\beta^{(k)}(\vec{x})\right]\prod_{\vec{x}\in B}\delta\left[\varphi_0^{(k)}(\vec{x})-\varphi_\beta^{(k)}(\vec{x})\right] \\
&\prod_{\vec{x}\in A_1}\delta\left[\varphi_0^{(k)}(\vec{x})-\varphi_\beta^{(k+1)}(\vec{x})\right]\prod_{\vec{x}\in A_2}\delta\left[\varphi_\beta^{(k)}(\vec{x})-\varphi_0^{(k+1)}(\vec{x})\right]\rho\left[\{\varphi_0^{(k)}(\vec{x})\},\{\varphi_\beta^{(k)}(\vec{x})\}\right]\Bigg\}.
\end{aligned}
$$

e.g. $\mathrm{tr}(\rho_{\mathrm{A}_1\cup\mathrm{A}_2}^{\mathrm{T}_{\mathrm{A}_2}})^3$



[Calabrese, Cardy, Tonni, 12]

$$\mathrm{tr}(\rho_{\mathrm{A}_1\cup\mathrm{A}_2}^{\mathrm{T}_{\mathrm{A}_2}})^3 = \frac{\mathcal{Z}_{3,2}}{\mathcal{Z}^3}$$

Partition function on a n-sheeted Riemann surface

A trick of computing the entanglement negativity

1. Trace norm

$$\mathrm{tr}|\rho_{A_1\cup A_2}^{T_{A_2}}| = \sum_i |\lambda_i| = \sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|$$

2. Momenta of the partially transposed reduced density matrix

$$\mathrm{tr}(\rho_{A_1\cup A_2}^{T_{A_2}})^n = \sum_i \lambda_i^n = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e}$$

$$= \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$$

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$$= \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$$

3. Entanglement negativity can be obtained by taking $n_e \to 1$

$$\mathcal{E}_{A_1 A_2} = \lim_{n_e\to 1} \ln \underline{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_2}\right)^{n_e}}$$

$$\frac{\mathcal{Z}_{n_e}}{\mathcal{Z}^{n_e}}$$

Entanglement negativity in quantum field theory

Partition function on a n-sheeted Riemann surface $\mathcal{R}_{n,N}$

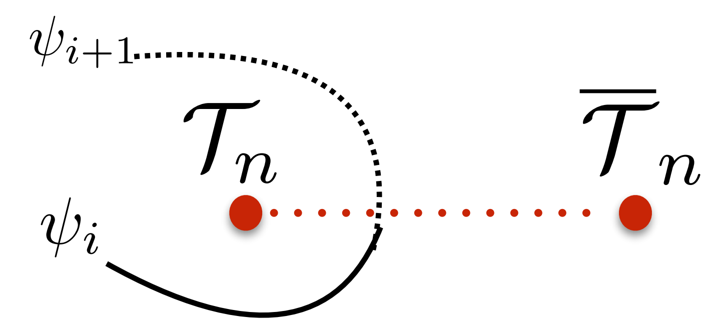
$$\mathcal{Z}_{n,N} = \int_{\mathcal{C}_{\mathrm{r}}} [d\psi_1 \cdots d\psi_n] \exp[-\int_C dz d\bar{z}(\mathcal{L}[\psi_1](z,\bar{z}) + \cdots + \mathcal{L}[\psi_n](z,\bar{z}))]$$

restricted path integral with:

Complex plane

$$\psi_i(x,0^+) = \psi_{i+1}(x,0^-) \qquad x \in A = \cup_{j=1}^{N} A_j, \quad j = 1, \cdots, N$$

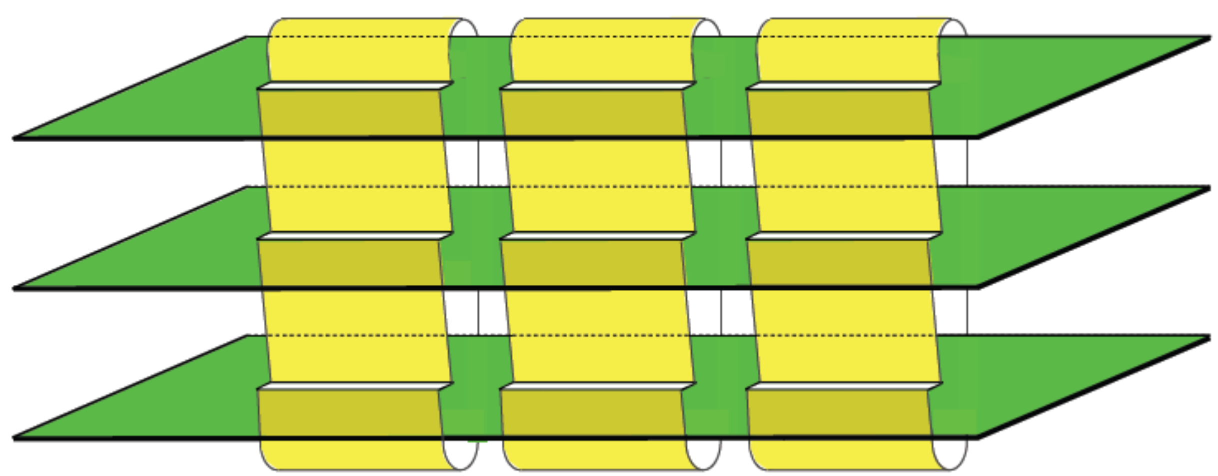
Define branch-point twist fields



$$\mathcal{T}_n := \mathcal{T}_\sigma, \quad \sigma : i \to i+1 \mod n$$

$$\overline{\mathcal{T}}_n := \mathcal{T}_\sigma^{-1}, \quad \sigma : i \to i-1 \mod n$$

$\mathcal{R}_{3,3}$



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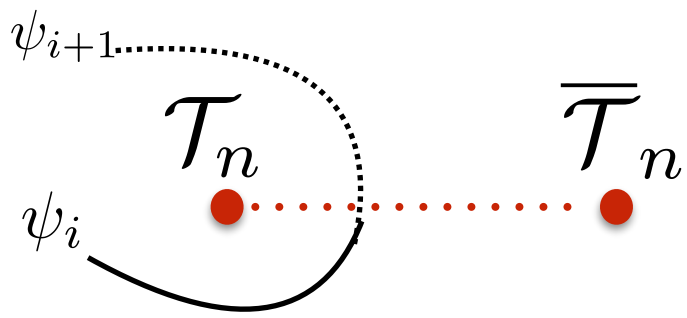


Correlation function of twist fields on a complex plane

$$\mathcal{Z}_{n,N} \propto \langle \mathcal{T}_n(u_1, 0) \overline{\mathcal{T}}_n(v_1, 0) \cdots \mathcal{T}_n(u_N, 0) \overline{\mathcal{T}}_n(v_N, 0) \rangle$$

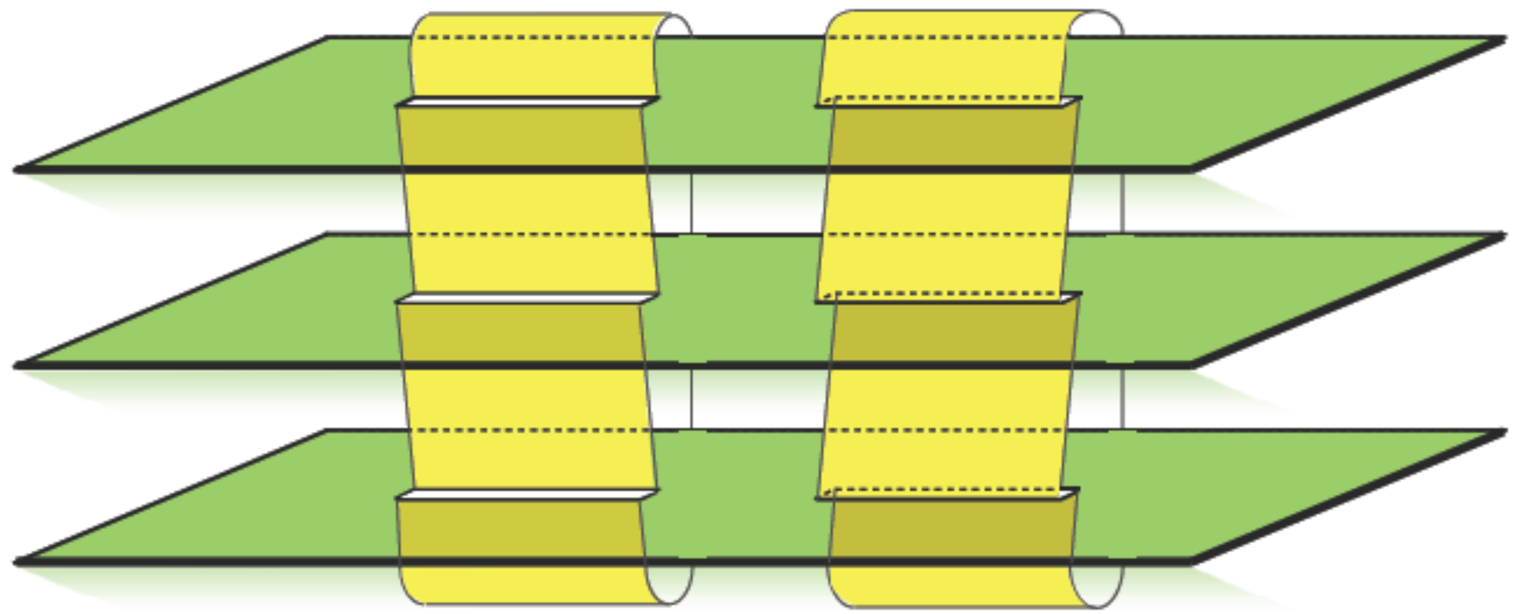
[Calabrese-Cardy 09]

Entanglement negativity in quantum field theory

Partial transposition

Gluing n copies of the above:

$$\mathrm{Tr}(\rho_A^{T_2})^n =$$



$$= \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle$$

[Calabrese-Cardy-Tonni, 12]

Entanglement negativity in quantum field theory
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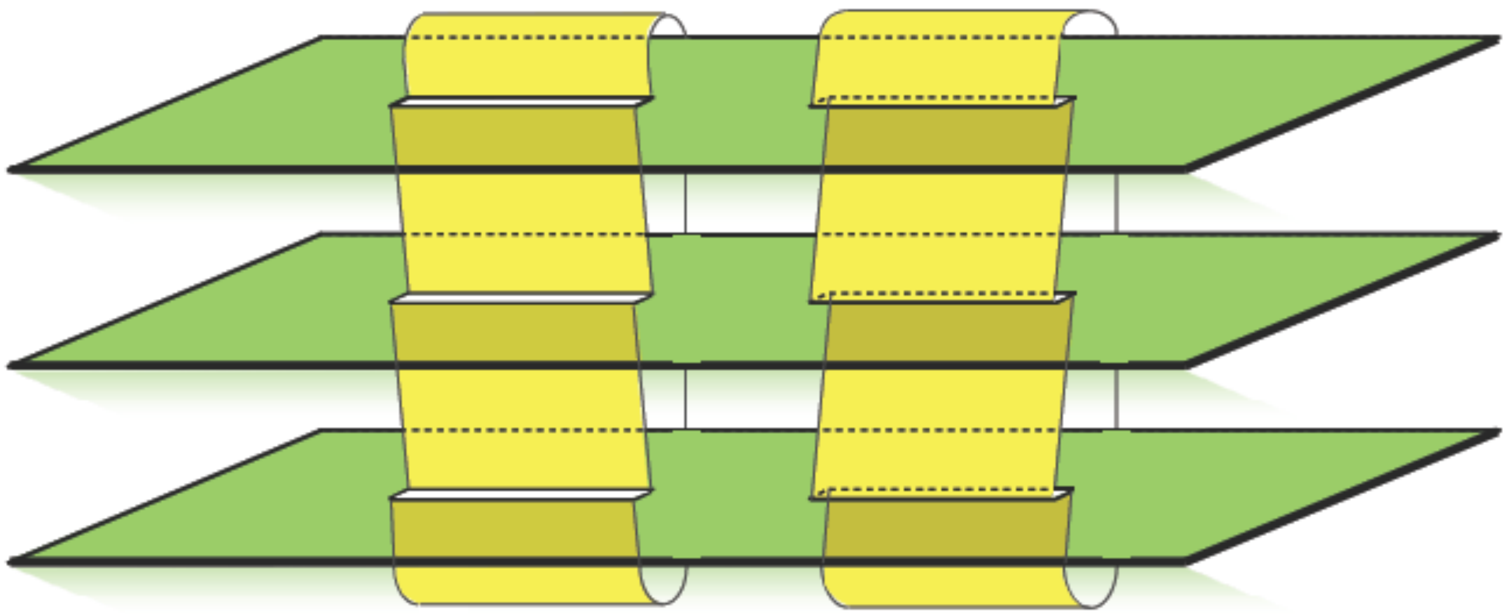


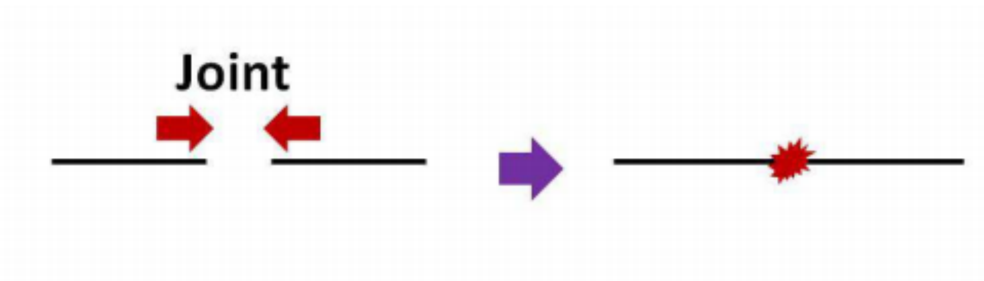
$$= \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle$$

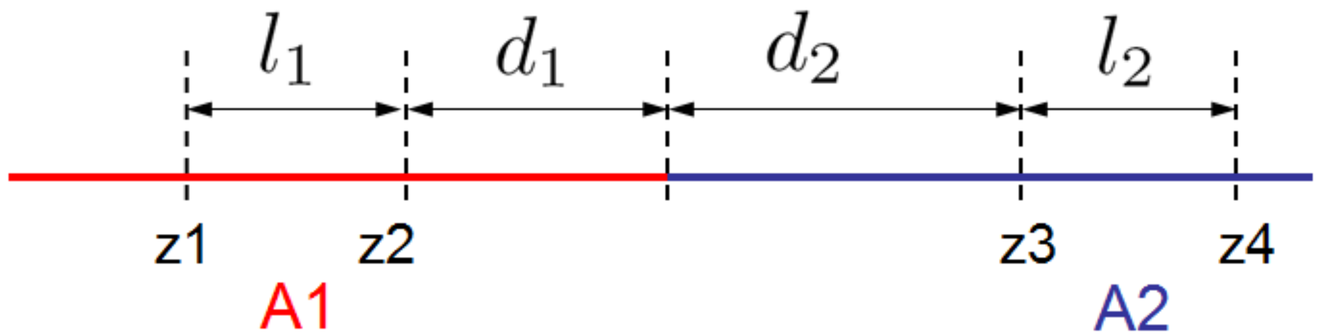
[Calabrese-Cardy-Tonni, 12]

Now we have enough ingredients!!!
Let us compute the entanglement negativity!!

Entanglement negativity after a local quench

[Wen, **PYC** and Ryu, 15]





$$\mathcal{E} = \lim_{n_e \to 1} \ln \left[\langle \mathcal{T}_{n_e}(z_1) \overline{\mathcal{T}}_{n_e}(z_2) \overline{\mathcal{T}}_{n_e}(z_3) \mathcal{T}_{n_e}(z_4) \rangle \right]$$

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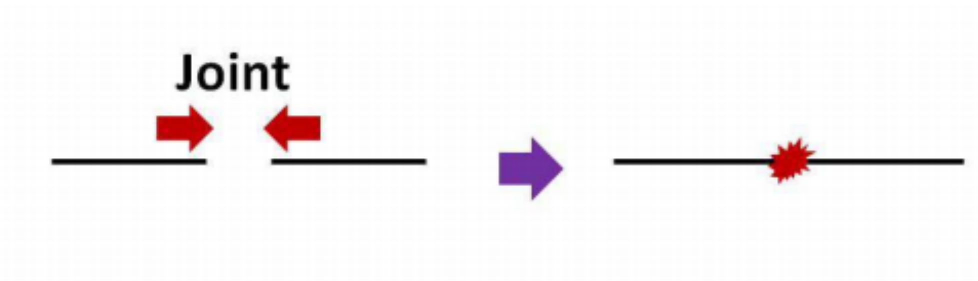

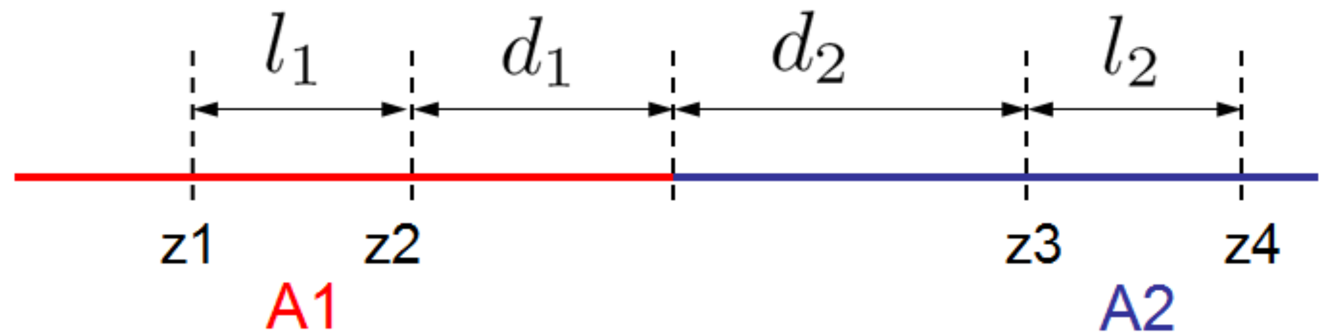

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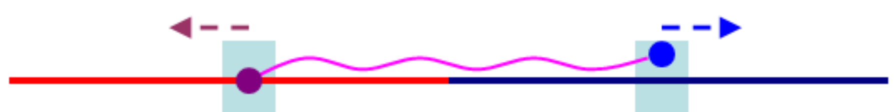
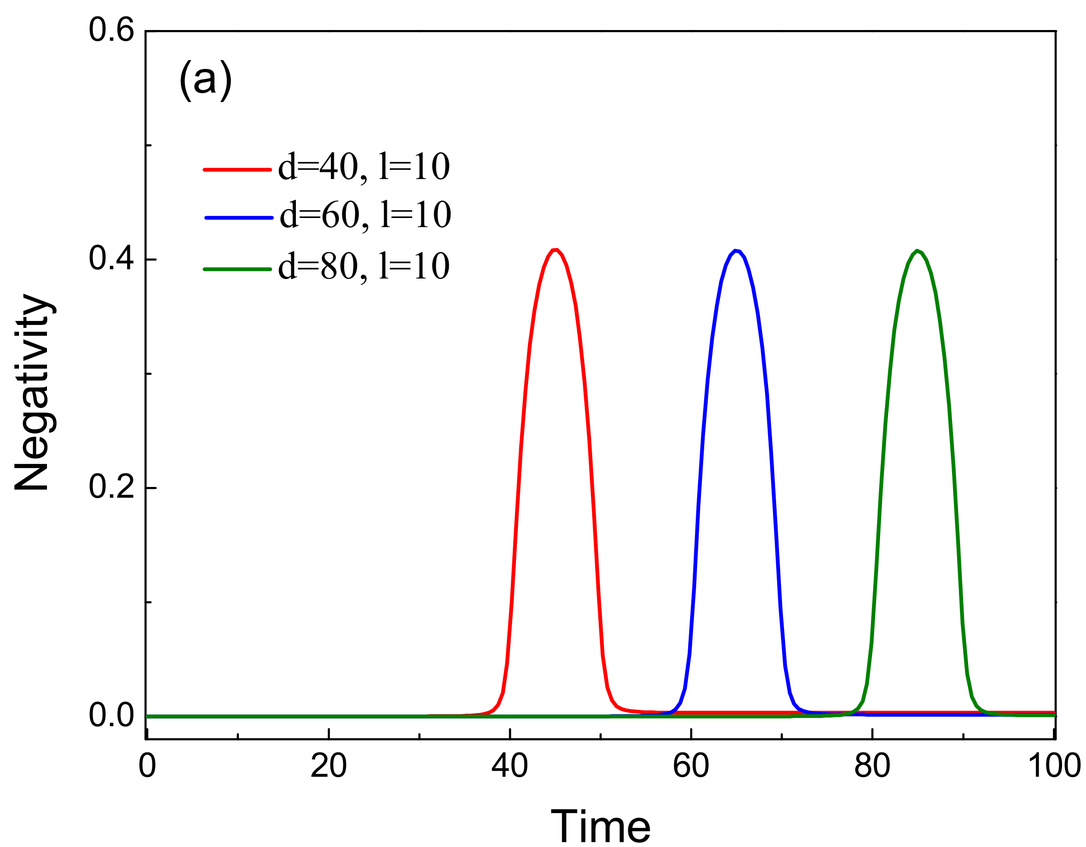
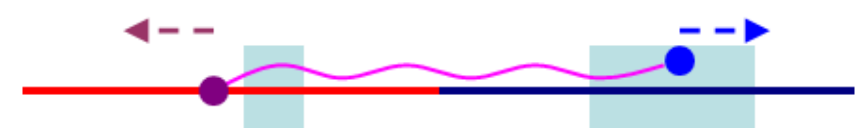
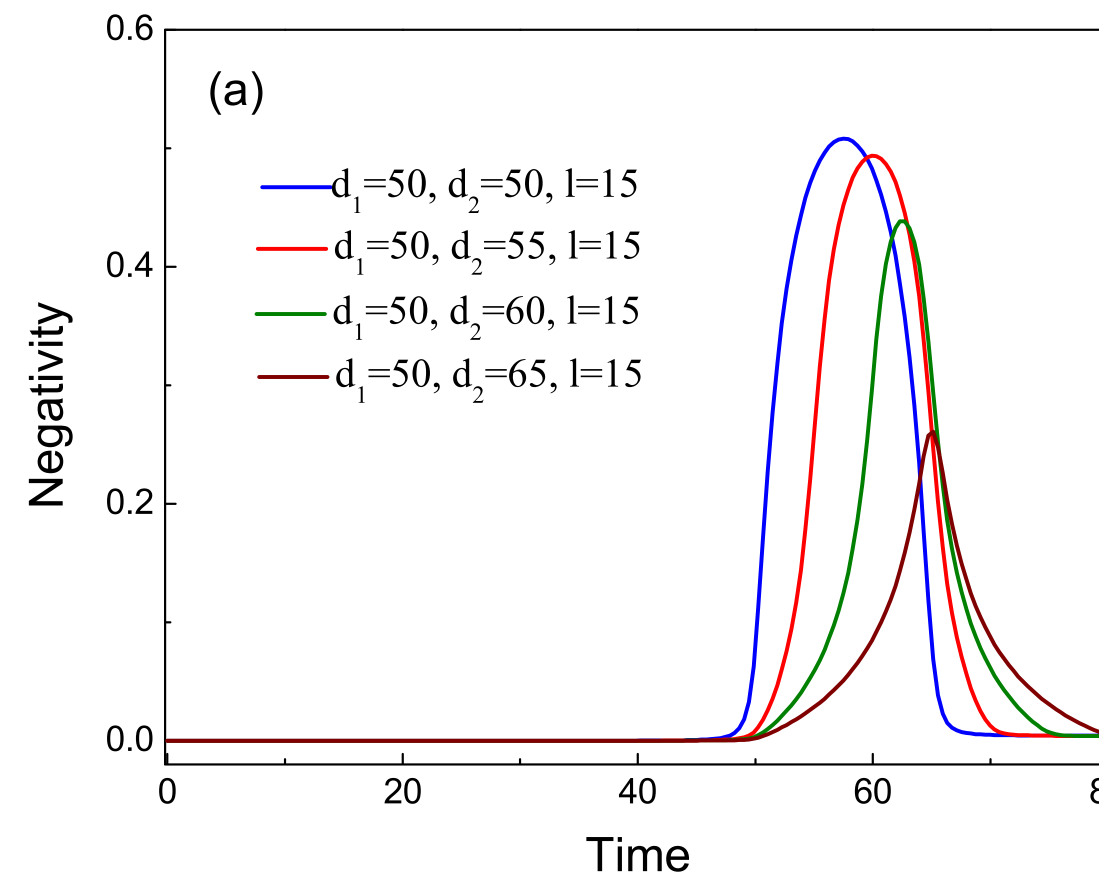
1. Two symmetric disjoint intervals





2. Two asymmetric disjoint intervals





- Entanglement negativity for free fermion—hard to compute
- Entanglement negativity for Conformal field theory—can measure the entanglement spread under quantum quenches
- Entanglement negativity for Chern-Simon theories can relate to geometry and topology

Motivation: The entanglement negativity for Chern-Simons theory is “topological”. And it relates to modulo S-matrix, which can related to anyon braiding.

Physics realization: fractional quantum Hall systems

Chern-Simons Theory

coupling constant (quantized)

1. CS theory

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Manifold

connection of a gauge group

2. Partition function

$$Z(M) = \int [\mathcal{D}A] \mathrm{e}^{iS_{\mathrm{CS}}(A)}$$

3. Wilson lines (links and knots)

$$W_R^{\mathcal{C}}(A) = \mathrm{tr}_R P \exp \int_{\mathcal{C}} A.$$

4. Correlators (partition function with links and knots)

$$Z(M, \hat{R}_1, \cdots, \hat{R}_N) = \langle W_{\hat{R}_1}^{\mathcal{C}_1} \cdots W_{\hat{R}_N}^{\mathcal{C}_N} \rangle = \int [\mathcal{D}A] \left(\prod_{i=1}^{N} W_{\hat{R}_i}^{\mathcal{C}_i} \right) \mathrm{e}^{iS_{\mathrm{CS}}}$$

Chern-Simons Theory

coupling constant (quantized)

1. CS theory [redacted]

group

2. Par[redacted]

You don't need these

3. Wil[redacted]

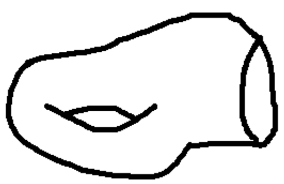
4. Correlators (partition function with links and knots)

$$Z(M, \hat{R}_1, \cdots, \hat{R}_N) = \langle W_{\hat{R}_1}^{\mathcal{C}_1} \cdots W_{\hat{R}_N}^{\mathcal{C}_N} \rangle = \int [\mathcal{D}A] \left(\prod_{i=1}^{N} W_{\hat{R}_i}^{\mathcal{C}_i} \right) \mathrm{e}^{iS_{\mathrm{CS}}}$$

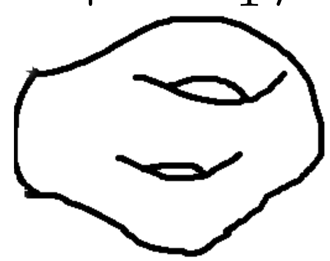
Minimun ingredients

1. The partition function can be computed from the canonical quantization of a CS theory on a 3-manifold with boundary.

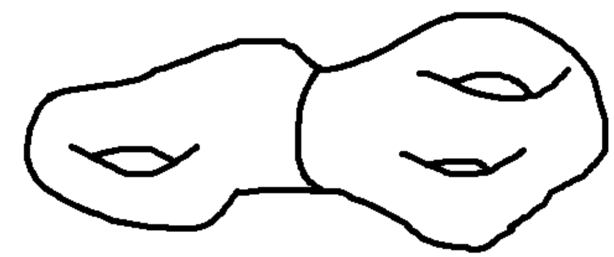
$\langle \Psi_{M_2} |$



$| \Psi_{M_1} \rangle$



$Z(M) = \langle \Psi_{M_2} | U_f | \Psi_{M_1} \rangle$



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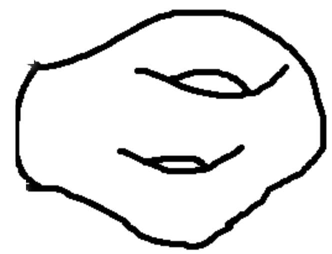

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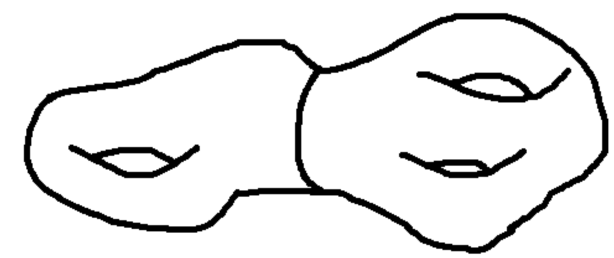


2. The partition function in the presence of Wilson lines

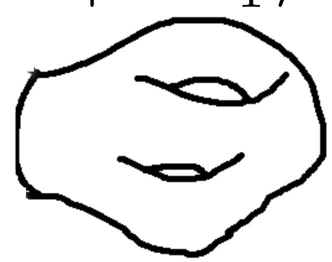
$$Z(S^2 \times S^1, \hat{R}_i, \hat{R}_j) = \langle \hat{R}_i | \hat{R}_j \rangle = \delta_{i,j}.$$

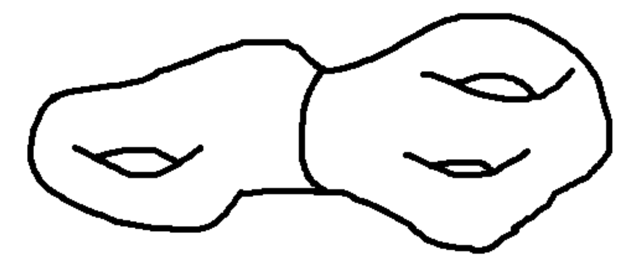
$$Z(S^3, \hat{R}_i, \hat{R}_j) = \langle \hat{R}_i | S | \hat{R}_j \rangle = \mathcal{S}_{ij}.$$

Minimun ingredients

1. The partition function can be computed from the canonical quantization of a CS theory on a 3-manifold with boundary.

$\langle \Psi_{M_2} |$ $\quad\quad$ $| \Psi_{M_1} \rangle$ $\quad\quad$ $Z(M) = \langle \Psi_{M_2} | U_f | \Psi_{M_1} \rangle$





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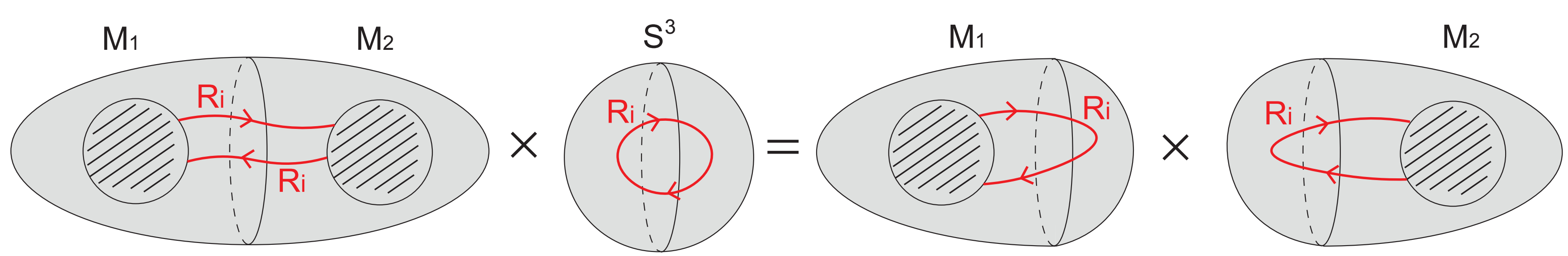
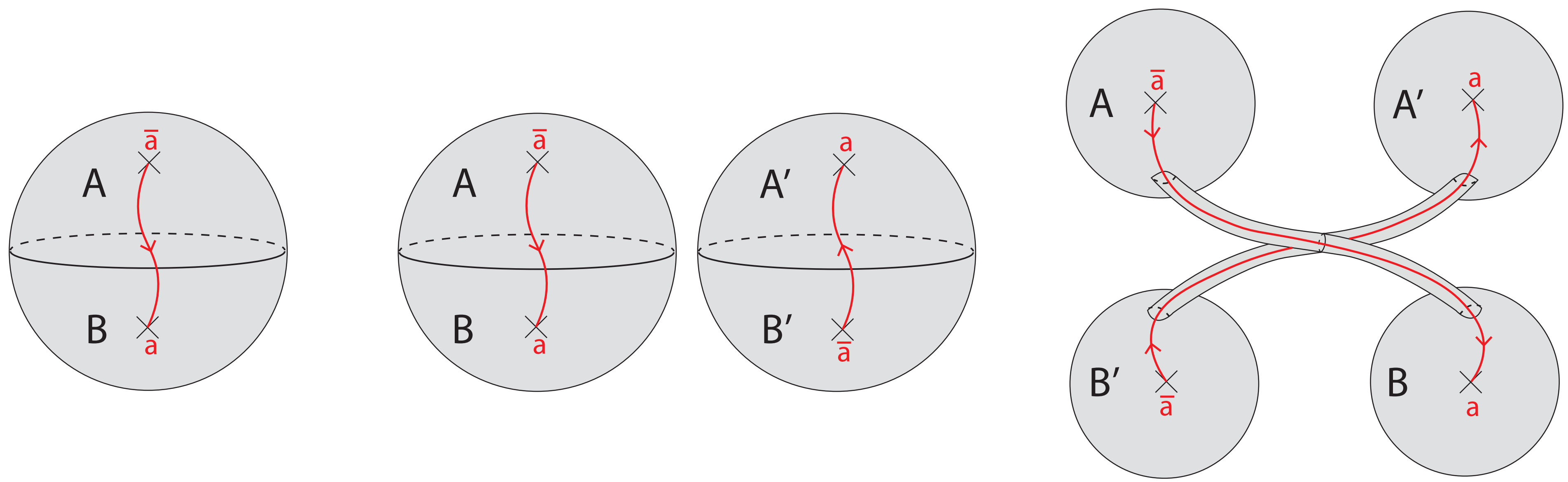
3. Factorability



$$Z(M, [\blacksquare_1, \blacksquare_2, \hat{R}_i, \overline{\hat{R}}_i]_{\mathcal{C}}) \cdot Z(S^3, \hat{R}_i) = Z(M_1, [\blacksquare_1, \hat{R}_i]_{\mathcal{C}_1}) \cdot Z(M_2, [\blacksquare_2, \hat{R}_i]_{\mathcal{C}_1})$$

Now let us compute the entanglement negativity in various cases

Ex1



(a) $|\Psi\rangle$

(b) $\rho_{A\cup B} = |\Psi\rangle\langle\Psi|$

(c) $\rho_{A\cup B}^{T_B}$

Now let us compute the entanglement negativity in various cases

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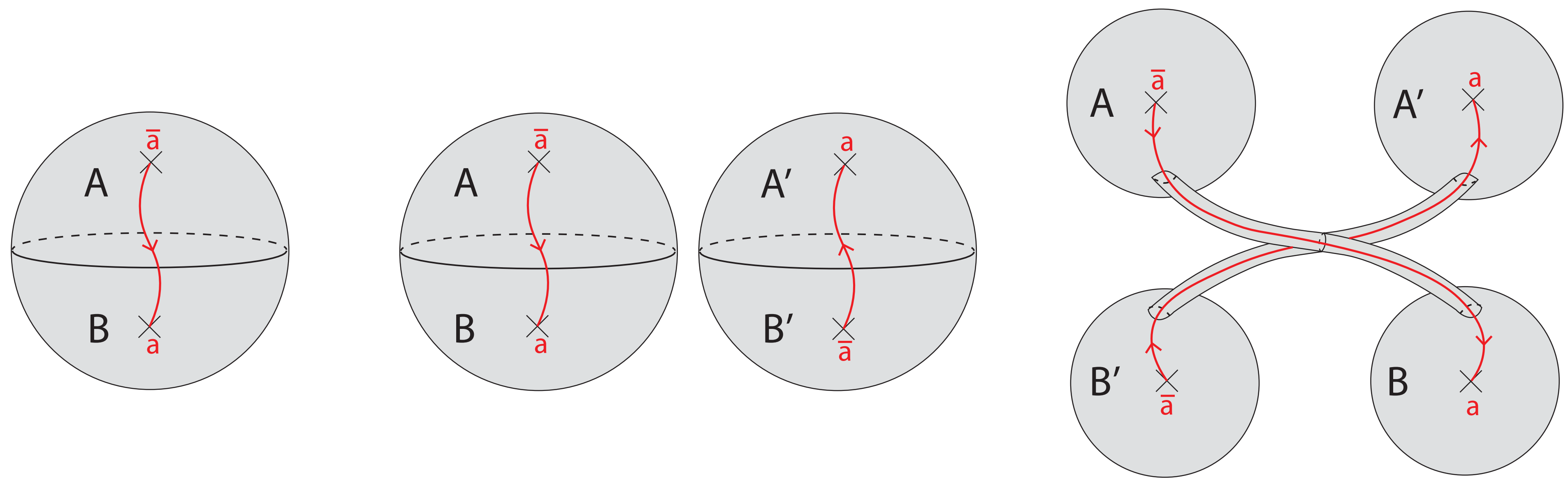

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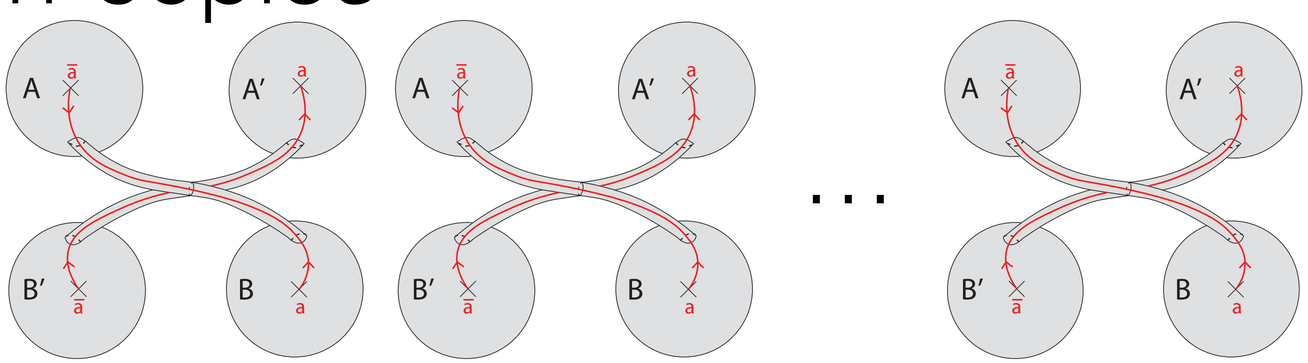
n-copies



$$\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_o}}{(\mathrm{tr}\rho^{T_B})^{n_o}} = \frac{Z(S^3,\hat{R}_a)}{Z(S^3,\hat{R}_a)^{n_o}} = Z(S^3,\hat{R}_a)^{1-n_o} = (\mathcal{S}_{0a})^{1-n_o}$$

$$\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_e}}{(\mathrm{tr}\rho^{T_B})^{n_e}} = \frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^{n_e}} = Z(S^3,\hat{R}_a)^{2-n_e} = (\mathcal{S}_{0a})^{2-n_e}$$

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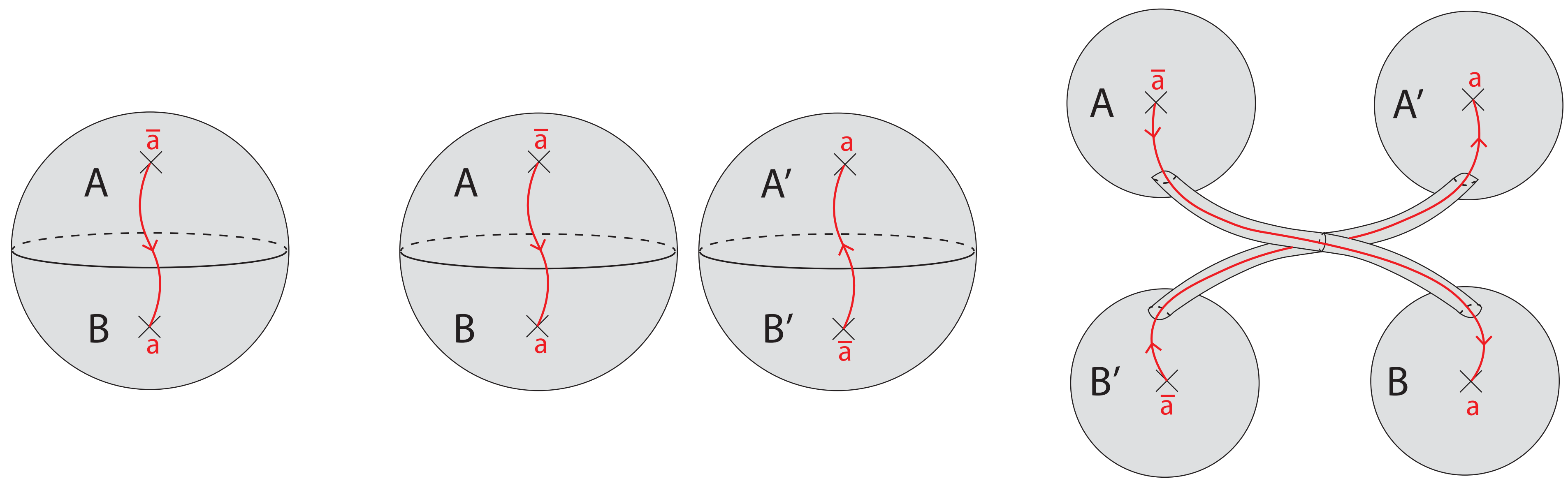

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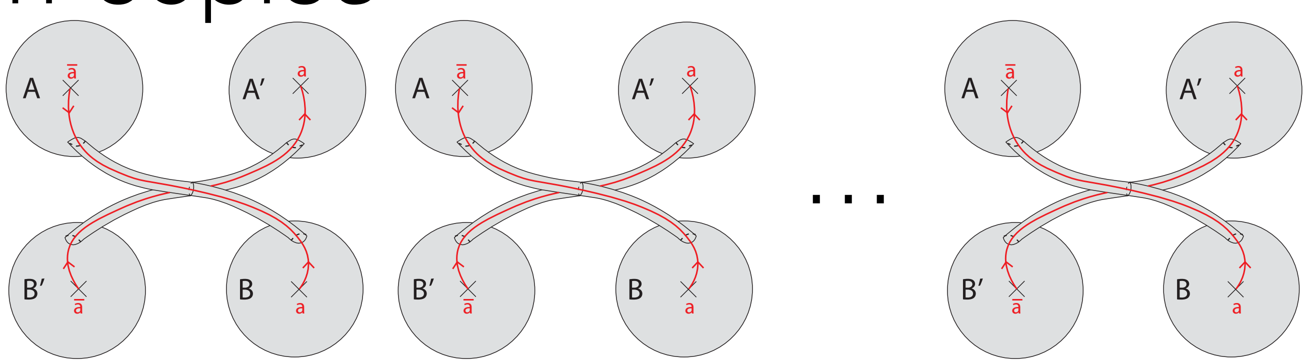
n-copies

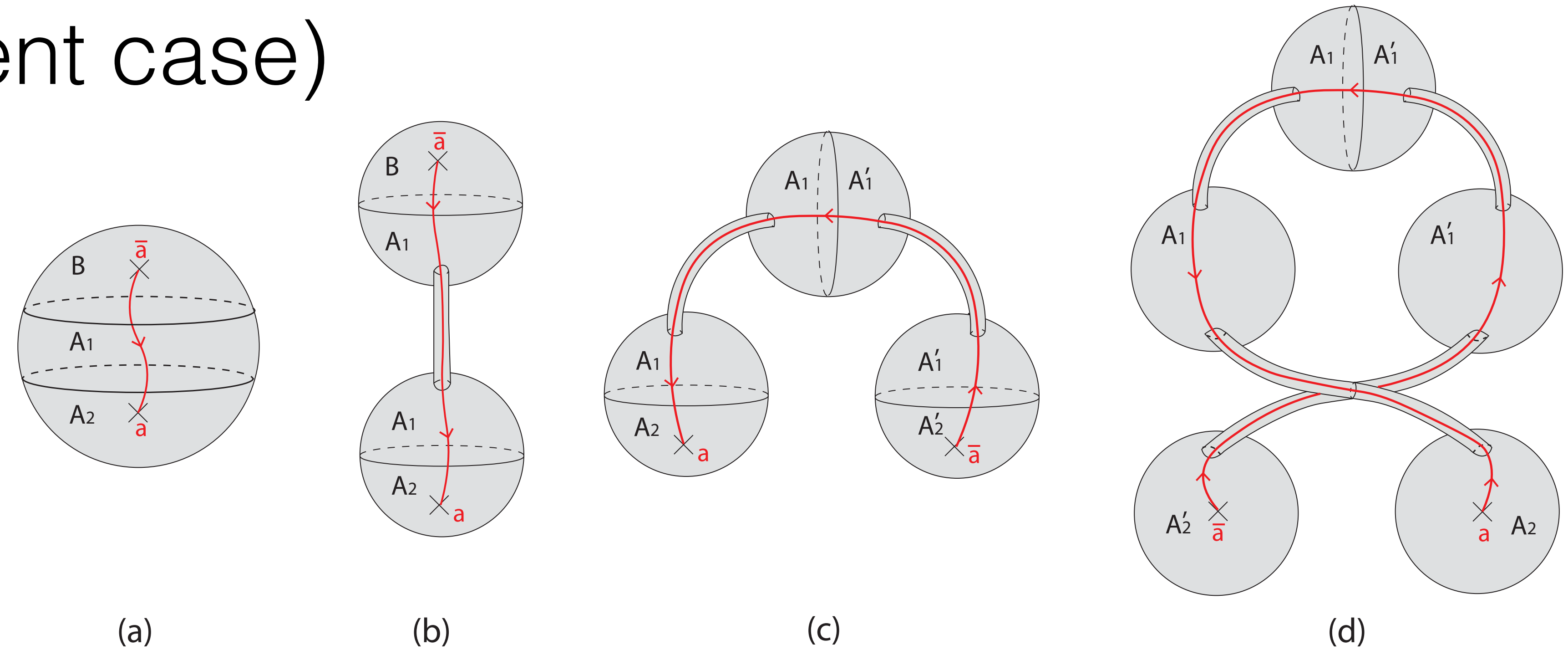


$$\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_o}}{(\mathrm{tr}\rho^{T_B})^{n_o}} = \frac{Z(S^3,\hat{R}_a)}{Z(S^3,\hat{R}_a)^{n_o}} = Z(S^3,\hat{R}_a)^{1-n_o} = (\mathcal{S}_{0a})^{1-n_o}$$

$$\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_e}}{(\mathrm{tr}\rho^{T_B})^{n_e}} = \frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^{n_e}} = Z(S^3,\hat{R}_a)^{2-n_e} = (\mathcal{S}_{0a})^{2-n_e}$$

$$\mathcal{E}_{AB} = \lim_{n_e\to 1} \ln \frac{\mathrm{tr}(\rho^{T_B})^{n_e}}{(\mathrm{tr}\rho^{T_B})^{n_e}} = \ln \mathcal{S}_{0a}$$

Ex2 (adjacent case)

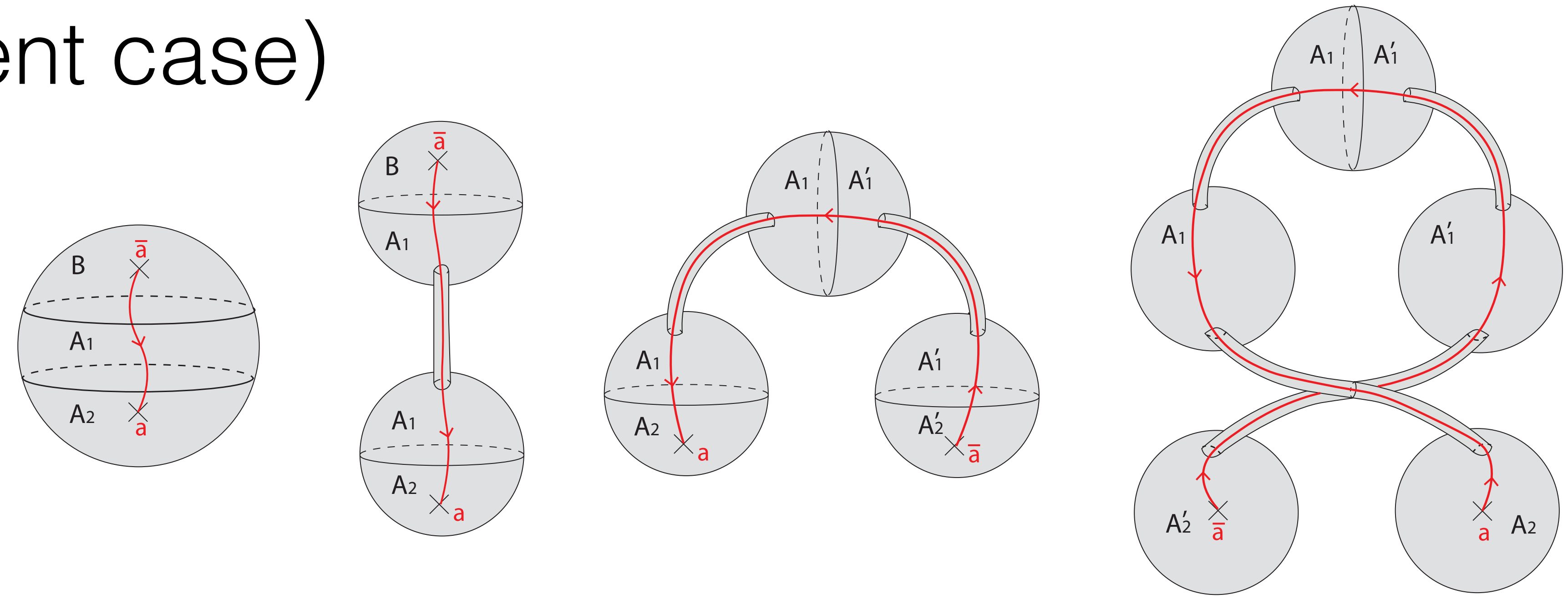


$|\Psi\rangle$

$\rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$

$\rho^{T_{A_2}}_{A_1\cup A_2}$

Ex2 (adjacent case)

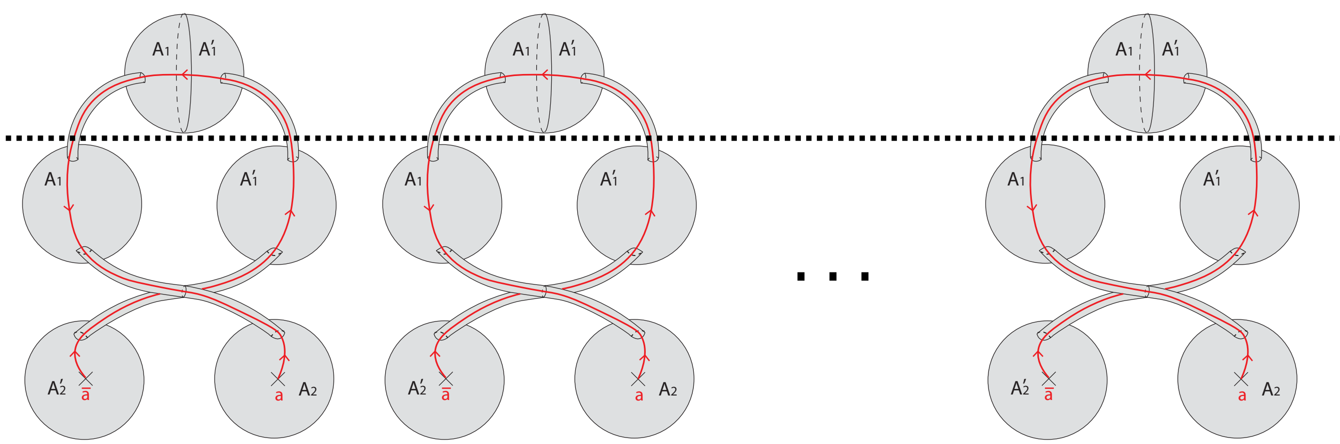


(a) (b) (c) (d)

$|\Psi\rangle$

$\rho_{A_1\cup A_2} = \mathrm{tr}_B |\Psi\rangle\langle\Psi|$

$\rho_{A_1\cup A_2}^{T_{A_2}}$

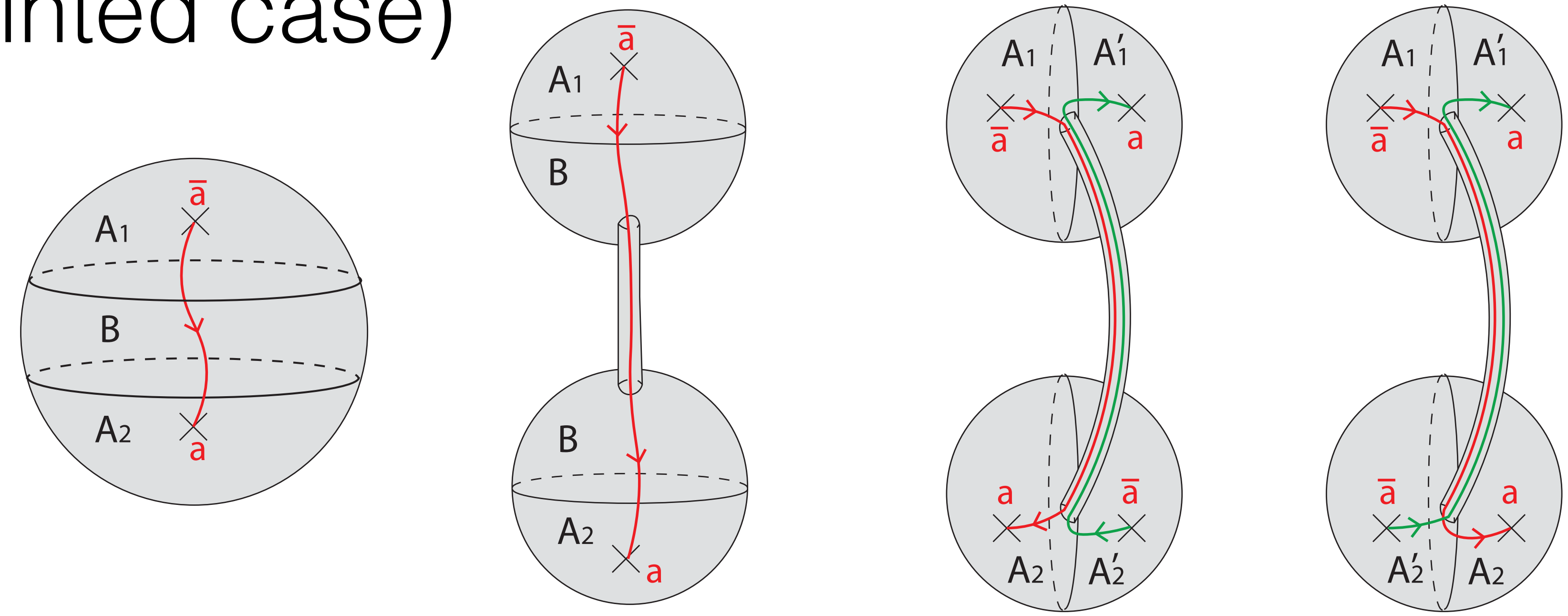


$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}} = \frac{1}{Z(S^3,\hat{R}_a)^{n_o}} \cdot \frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^{n_o}} = Z(S^3,\hat{R}_a)^{2-2n_o} = (\mathcal{S}_{0a})^{2-2n_o}$$

$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}} = \frac{1}{Z(S^3,\hat{R}_a)^{n_e}} \cdot \frac{Z(S^3,\hat{R}_a)^3}{Z(S^3,\hat{R}_a)^{n_e}} = Z(S^3,\hat{R}_a)^{3-2n_e} = (\mathcal{S}_{0a})^{3-2n_e}$$

$$\mathcal{E}_{A_1A_2}(B\neq\emptyset) = \mathcal{E}_{A_1A_2}(B=\emptyset).$$

Ex3 (disjointed case)



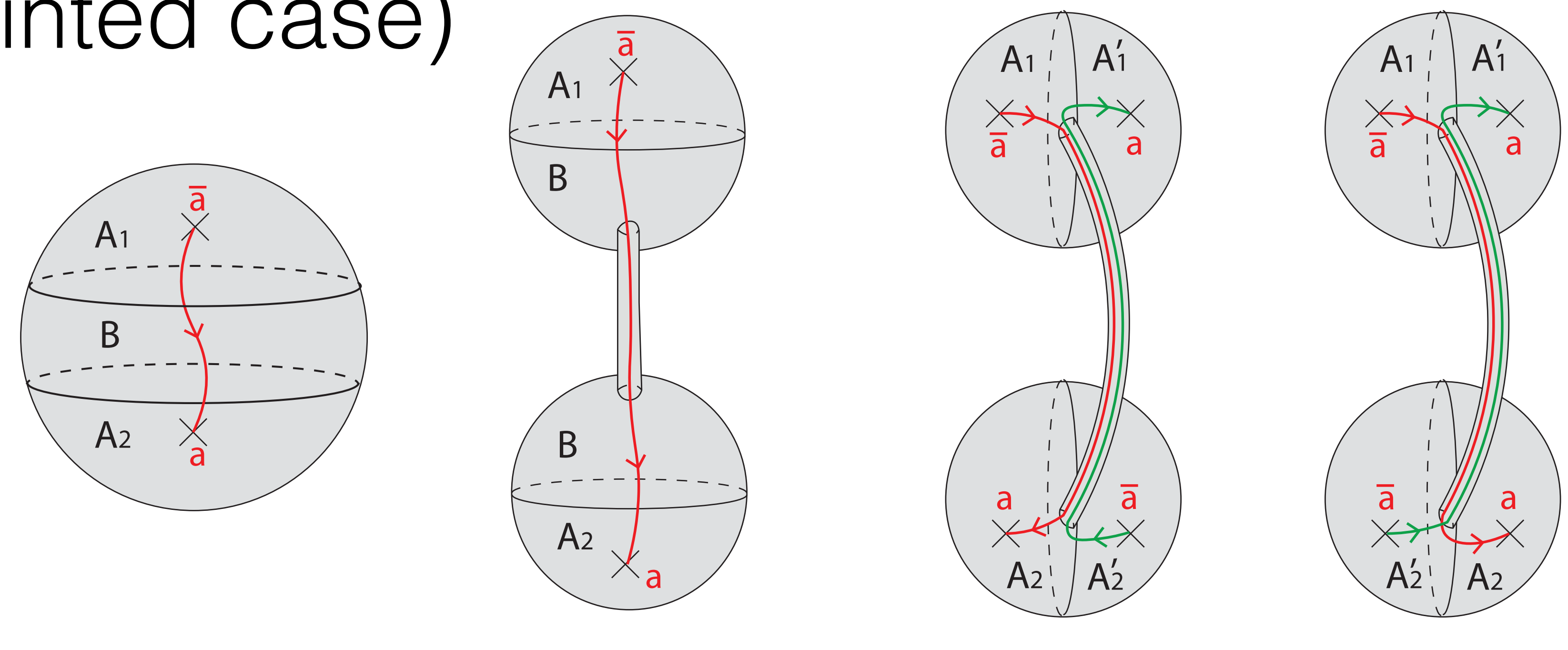
(a) (b) (c) (d)

$|\Psi\rangle$

$\rho_{A_1 \cup A_2} = \mathrm{tr}_B |\Psi\rangle\langle\Psi|$

$\rho_{A_1 \cup A_2}^{T_{A_2}}$

Ex3 (disjointed case)



(a) $|\Psi\rangle$

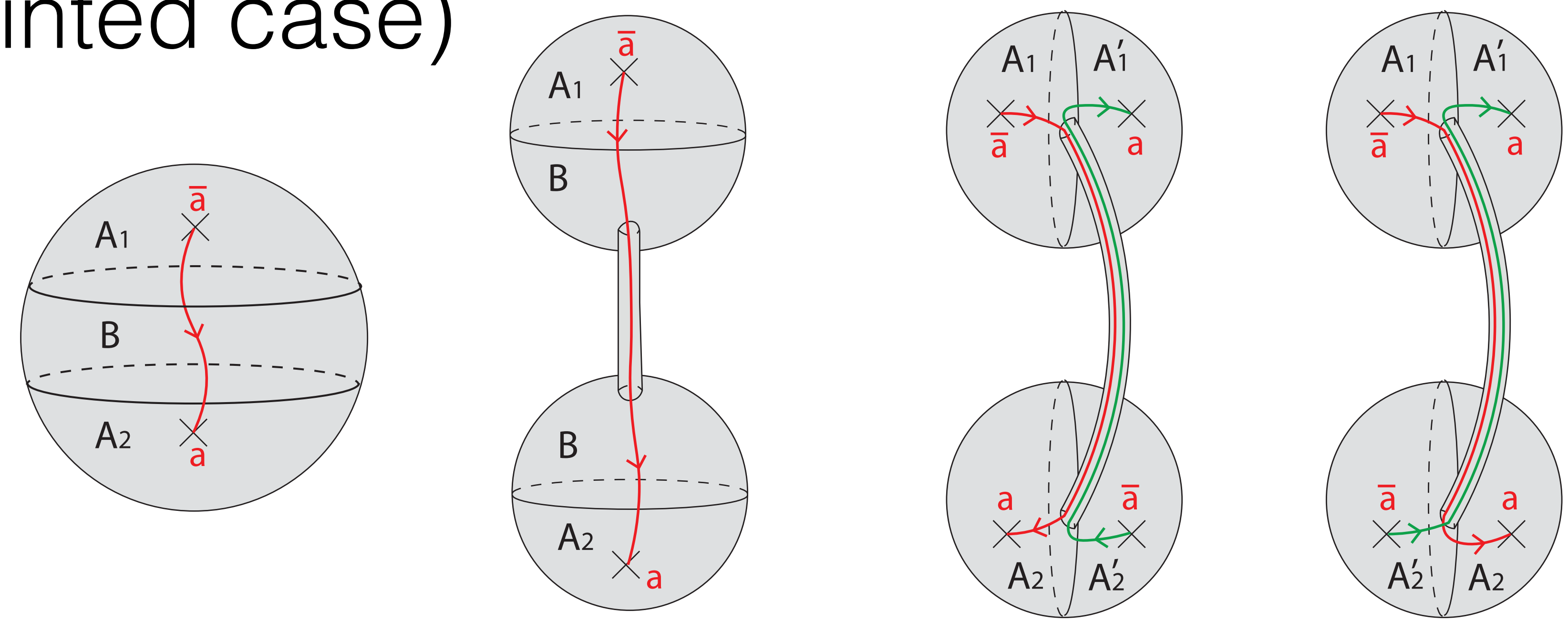
(b)

(c) $\rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$

(d) $\rho_{A_1\cup A_2}^{T_{A_2}}$

$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^n}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^n} = \frac{1}{Z(S^3,\hat{R}_a)^n}\cdot\frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^n} = Z(S^3,\hat{R}_a)^{2-2n} = (\mathcal{S}_{0a})^{2-2n}$$

Ex3 (disjointed case)



(a) (b) (c) (d)

$|\Psi\rangle$

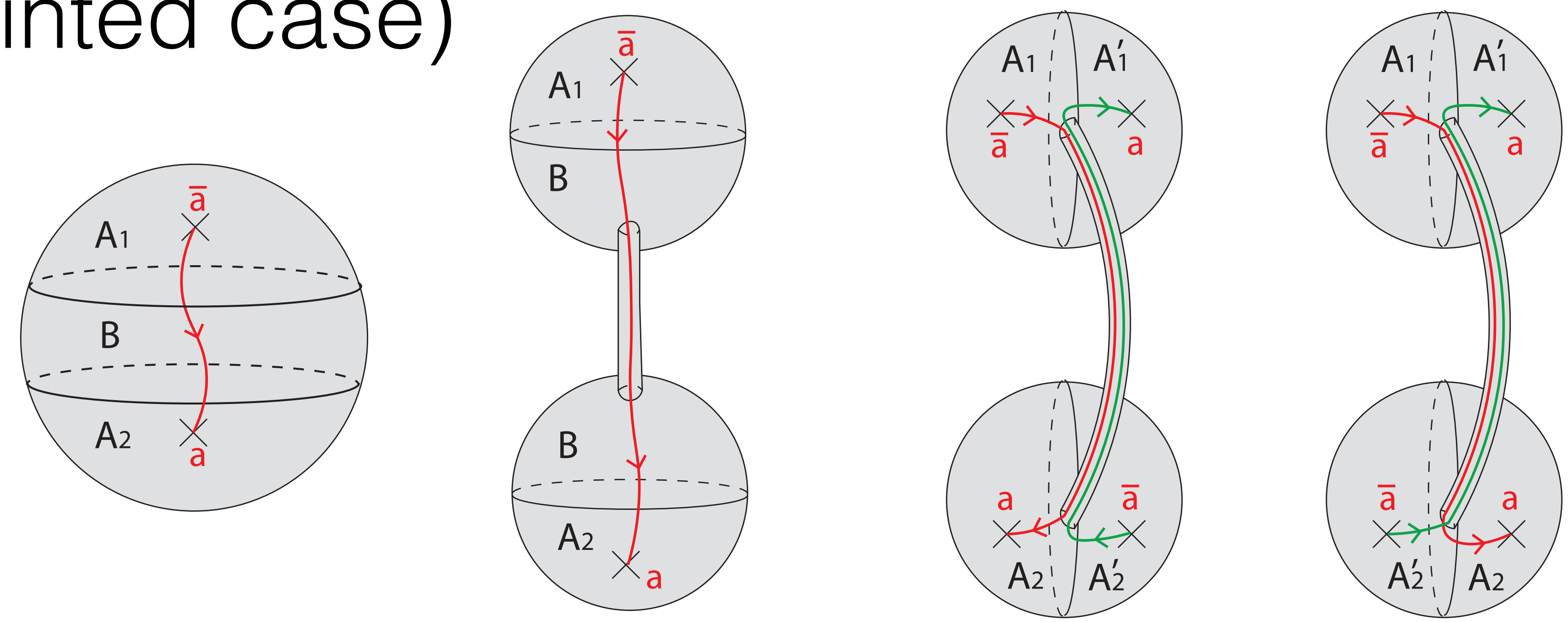
$\rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$

$\rho^{T_{A_2}}_{A_1\cup A_2}$

$$\frac{\mathrm{tr}\left(\rho^{T_{A_2}}_{A_1\cup A_2}\right)^n}{\left(\mathrm{tr}\rho^{T_{A_2}}_{A_1\cup A_2}\right)^n} = \frac{1}{Z(S^3,\hat{R}_a)^n}\cdot\frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^n} = Z(S^3,\hat{R}_a)^{2-2n} = (\mathcal{S}_{0a})^{2-2n}$$

$$\mathcal{E}_{A_1A_2} = \lim_{n_e\to 1}\ln\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_e}}{\left(\mathrm{tr}\rho^{T_B}\right)^{n_e}} = \ln\left(\mathcal{S}_{0a}\right)^0 = 0.$$

Ex3 (disjointed case)



(a) (b) (c) (d)

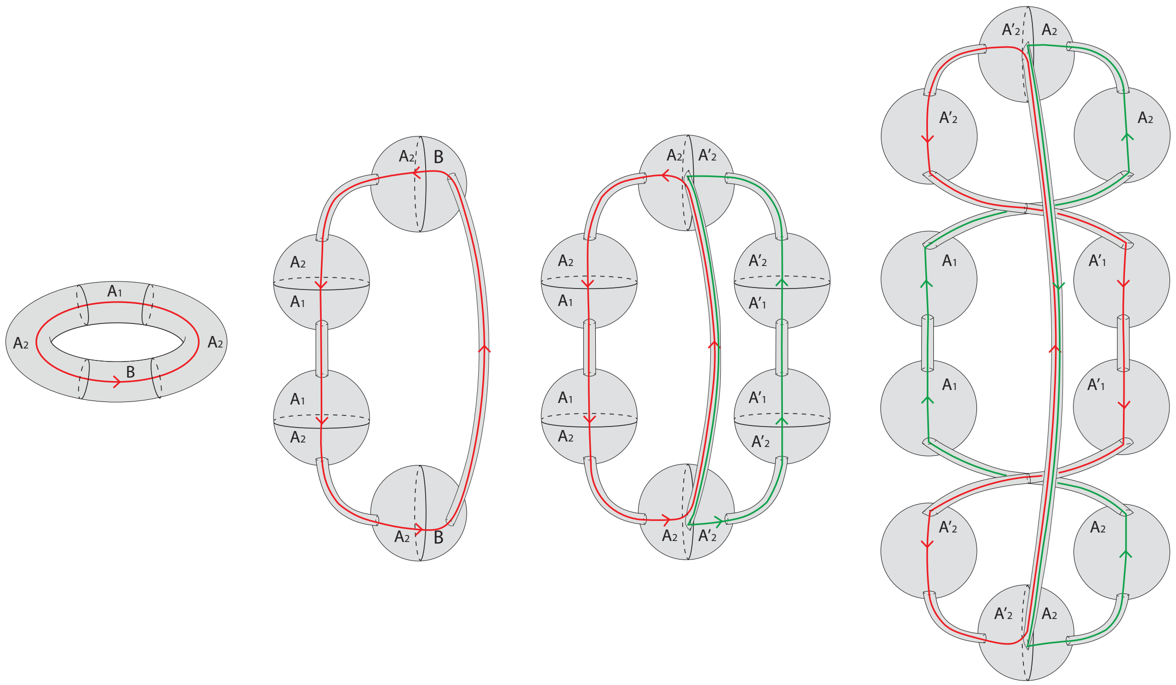
$|\Psi\rangle$ $\qquad \rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$ $\qquad \rho^{T_{A_2}}_{A_1\cup A_2}$

$$\frac{\mathrm{tr}\left(\rho^{T_{A_2}}_{A_1\cup A_2}\right)^n}{\left(\mathrm{tr}\rho^{T_{A_2}}_{A_1\cup A_2}\right)^n} = \frac{1}{Z(S^3,\hat{R}_a)^n}\cdot\frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^n} = Z(S^3,\hat{R}_a)^{2-2n} = (\mathcal{S}_{0a})^{2-2n}$$

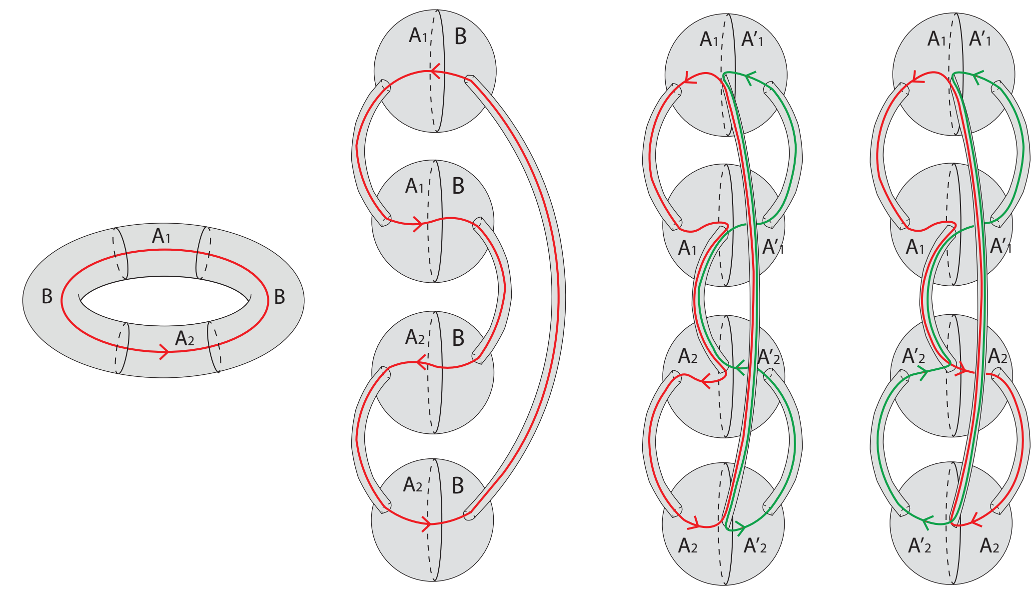
$$\mathcal{E}_{A_1A_2} = \lim_{n_e\to 1}\ln\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_e}}{(\mathrm{tr}\rho^{T_B})^{n_e}} = \ln(\mathcal{S}_{0a})^0 = 0.$$

No entanglement if A1 and A2 do not have interfaces!

More cases



(a) (b) (c) (d)



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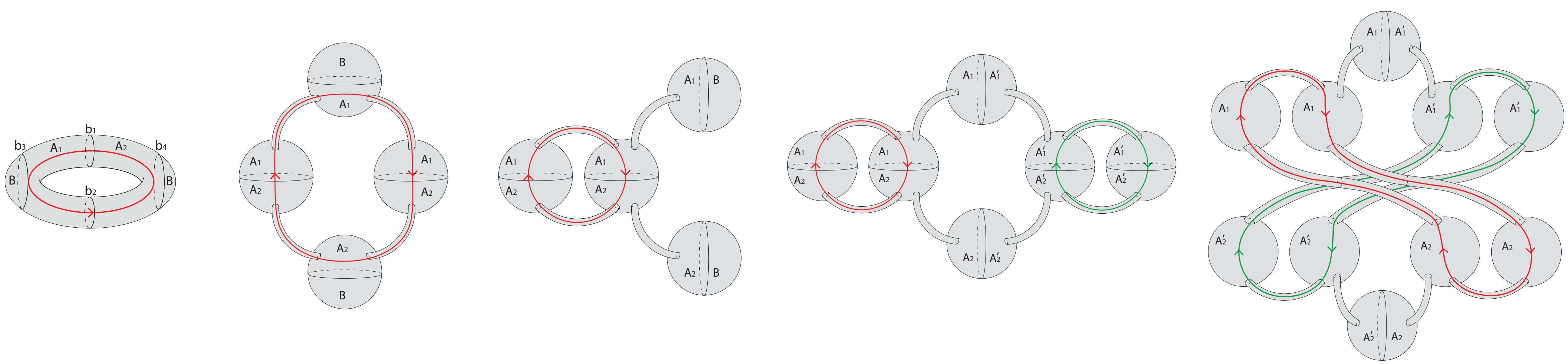

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Conclusion and future directions:

- Entanglement negativity is a very useful tool and links to dynamics, topology and geometry.
- Generalization for higher dimensions?
- Generalization other topological field theories?
- What is the holographic picture for entanglement negativity??
- Evolution of of the entanglement negativity for other quenches?